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UNIVERSITY COLLEGE, UNIVERSITY OF LONDON.

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TECHNICAL SERIES, III.

ON THE GRAPHICS OF METAL ARCHES,

WITH SPECIAL REFERENCE TO THE RELATIVE STRENGTH OF TWO-PIVOTED, THREE-PIVOTED AND BUILT-IN METAL ARCHES.

BY

L. W. ATCHERLEY,
DEMONSTRATOR IN APPLIED MECHANICS, UNIVERSITY COLLEGE, LONDON,

AND

KARL PEARSON, F.R.S.,
PROFESSOR OF APPLIED MATHEMATICS AND MECHANICS, UNIVERSITY COLLEGE, LONDON.

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On the Graphics of Metal Arches, with Special Reference to the Relative Strength of Two-pivoted, Three-pivoted and Built-in Metal Arches.

By L. W. Atcherley *and* Karl Pearson, *F.R.S.*

(1.) *Introductory.*

The object of this paper is twofold. There exists, as far as we are aware, no complete account in English of the graphical treatment of metal arches.* Probably the best foreign account is that due to M. Maurice Lévy in his 'La Statique Graphique,' tome III., pp. 1–183. That account is not only in several respects insufficient, but it seems to us to suffer to some extent from being too theoretical. We do not feel convinced that the writer has actually carried out on the drawing-board the constructions he describes for any considerable number of large arches, and so experienced the real difficulties which arise in graphical practice. During the last 20 years annual courses on metal arches have been given in the Department of Applied Mathematics at University College, and in the course of the drawing practice the stresses and deflections of a considerable number of British and continental metal arches have been actually determined graphically. This experience has taught us one point, which we think would not have been overlooked by those English text-book writers, who give a few meagre pages to the graphical treatment of arches, had they really put their treatment into practice: namely, using the largest available drawing-board and exaggerating the vertical ordinates to any workable extent, it is impossible to obtain results sufficiently correct even for practical use by purely graphical processes. For all the "flat" arches of modern construction a mixture of arithmetic and graphics is needful. What we may term a *semi-graphical* treatment is absolutely necessary. A purely graphical treatment may possibly suffice for a semi-circular or semi-elliptic arch, as we have ourselves found for the arches at Brest, or the rib of the station roof at Turin; it fails completely when applied to any arch with as small a ratio of rise to span as those of the St. Louis Bridge or the Pont Alexandre III. Our first aim, therefore, will be to give the mathematical theory of arches in a brief and fairly elementary form, and deduce from this theory the combined graphical and arithmetical treatment which we have found effective in practice. By "effective in practice" we do not mean a construction which a draughtsman can take up light-heartedly and complete in an afternoon. Nor is this needful; any structure of this kind is a great and important work, and means days, if not weeks, of labour even for a first-class draughtsman. Such labour, however, is a small item in the actual cost of construction, and is fully admitted to be a necessary part of such large designs. In academic teaching in the higher technical schools it is impossible to practise the average student in an arch of each type, but half a term at least given to the theory with practical drawing work on one type has been found in our experience to give a sound basis for later practical work.

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* The fundamental memoir is that by H. T. Eddy, 'Neue Constructionen aus der graphischen Statik,' Leipzig, 1880. This is suggestive, but far from complete, especially as to the closeness of the approximations. He is followed by Lévy. We are not acquainted with any other important works on the subject, and the present paper is essentially the publication of the many years' drawing-office practice at University College.

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Although variants can be imagined and do actually occur, there are four fundamental types of arches or ribs that deserve special consideration :—

(i). The arch with its terminals hinged or pivoted. We shall speak of this as the two-pivoted arch. As an example we may cite the Pimlico Railway Bridge, where the semi-circular ends of the ribs work in cast-iron sockets of the same form.

(ii.) The arch with its terminals built in. This type is well represented by the bridge at St. Louis in America, where the terminals are fixed into solid masses of masonry.

(iii.) The arch pin-jointed at the crown and at the terminals. The newest type of this is provided by the ribs of the Pont Alexandre III. in Paris, with actual pins at the crown and terminals.

A variant of this, with the terminals built in, but the pin-joint still at the crown, and two further pin-joints, one on either side between the terminals and the crown, might give theoretical advantages and perhaps be of some practical service.

(iv.) The arch in which the pin-jointed terminals are connected by a metal tie. This type is, we believe, that of the ribs of St. Pancras Railway Station, the ties being below the level of the station hall.

We shall give the theory and graphical constructions for these four cases.

Our second, and possibly chief, aim is to consider the relative merits of the first three types by the aid of graphical constructions for the same-dimensioned arch treated as belonging successively to these types. It must be at once admitted that this is a theoretical investigation; local circumstances, or special conditions as to appearance or cost, may dictate the use of one or other type. But quite apart from this, we believe that there are very few engineers to-day who would be able to say whether a two-pivoted, or a three-pivoted, or a built-in arch of the same dimensions, to carry a given load, would, having regard to actual stresses, temperature stresses, and extent of deflections, be the stronger and more effective structure. Of course, one man, with one or other end in view, would apportion different weights to dead and live load stresses, temperature stresses, and dead and live load deflections. He might not conclude that one or other type was definitely the better. Still, any data upon which to form even a relative judgment seem at present so completely wanting that the actual treatment in an individual instance of the three types, besides forming an illustration of the semi-graphical processes, cannot fail to throw some light on a problem, which must have arisen in the mind of more than one engineer, and for the answer to which he would, as far as our knowledge goes, have sought in vain in practical and theoretical treatises: What would be the relative advantages of a two-pivoted, three-pivoted, and doubly built-in arch? We can of course give no final answer, but we hope to have provided material to assist the judgment of such a man in future.

(2.) *Theoretical Preliminaries.*

We make the following assumptions:—

(i.) The loading lies in, or is symmetrical about, one plane which contains the central line (line of centroids of the cross-sections) of the arch or rib.

(ii.) The cross-sections are perpendicular to the plane of loading, and a principal axis of each cross-section lies in this plane.

(iii.) The radius of curvature of the arch or rib (*i.e.*, that of its central line) at each point is large compared with the linear dimensions of the cross-section.

(iv.) The cross-section, if it changes its dimensions at all, only does so gradually and continuously throughout the length of the arch.

The following are the usual definitions:—

Let the resultant of all the forces to the right, say, of any cross-section consisting of loads and reactions be F, then:

(α.) The component Q of F, perpendicular to the cross-section, is *the total thrust on the cross-section.*

(β.) The component S of F, parallel to the cross-section, is *the total shear on the cross-section.*

(γ.) The moment M of F, round the central axis of the cross-section, *i.e.*, a line through the centroid perpendicular in this case to the plane of loading, is *the bending moment at the cross-section.*

(δ.) The envelope of the forces F is *the line of pressure of the arch.*[*]

(ϵ.) Supposing the two terminals on one level and the whole load vertical, as we shall suppose in this paper, then the horizontal component H of the total reaction at either terminal is the *horizontal thrust of the arch.*

Elementary Considerations.

(a.) So soon as the reactions at the points of support have been found, Q, S, and M can be determined by the ordinary processes of resolution of elementary graphical statics.

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[*] A slip was made in the definition of line of pressure in 'Drapers' Research Memoirs,' II., p. 2. The line of pressure is always the envelope of the resultant force on a cross-section; the line of resistance is the locus of the load points of the cross-sections. For an arch it is convenient to draw the former, for a dam the latter.

(*b.*) Whether the terminals be supported or built in, the *vertical* reactions must be in equilibrium with the whole load, which we suppose vertical. Hence they must be absolutely identical with the reactions which we should find in the case of a simple beam of the same span carrying the same load, and they can therefore be found at once by the same easy link- and vector-polygon construction.

(*c.*) At a pivoted terminal the horizontal component of the reaction is the unknown thrust of the arch H, the vertical component is the same as for a simply supported beam of the same loading.

(*d.*) At a built-in terminal the reactional forces reduce to a single force and a couple; the single force may be resolved into a horizontal force H and a vertical force which is equal to the "beam reaction." By altering the components of the couple to H, without altering its moment, we can balance by one of the components the above horizontal force H. We are thus left with the vertical reaction and a horizontal force H acting at some distance above the terminal. This horizontal force H will be spoken of as the thrust of the arch in the case of built-in terminals. Let the two terminal vertical reactions be R_A and R_B acting at the terminals A and B respectively.

(*e.*) Let a vector (or force) polygon be constructed by plotting H horizontally, then R_A vertically, then in order the loads on the arch between A and B, then R_B vertically upwards and $-H$ horizontally. We must, since there is equilibrium, return to the point at which we started. Now, using this starting point as pole, construct a link (or funicular) polygon for the loads on the arch, starting the first link through the point on the space diagram in which the horizontal thrust H meets the vertical reaction R_A. A little consideration shows that this link polygon is, whether the terminals be built in or pivoted, the line of pressure defined under (δ) above.

(*f.*) The forces on the arch consist of the loads, the vertical reactions, and the horizontal thrusts at A and B. The former are exactly the same as in the case of a simple beam, and the bending moment due to them is measured in the same way, *i.e.*, is the product of the vertical intercept on the link polygon at any cross-section into the polar distance, *i.e.*, the horizontal thrust. The moment of the latter is the product of the horizontal thrust into its distance from the central line of the arch. Hence combining, we have at once the theorem,* known as that of Eddy, that: *the bending moment at any point of an arch is the product of the horizontal thrust of the arch into the vertical intercept at the point between the line of pressure and the central line.*

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* Another and shorter proof is as follows:—Produce the vertical line at any cross-section to meet the line of pressure, the tangent to the latter gives the position of the resultant force on the cross-section. Resolve this resultant force vertically and horizontally, the latter component is the thrust of the arch, and alone has moment about the centroid, but its distance from the centroid is the vertical intercept of EDDY'S theorem, whence that theorem follows. Care must be taken not to confuse the above point of resolution with the load-point of the cross-section of the arch, which is the point where the above tangent or resultant force meets the cross-section.

Thus the discovery of the bending moment is thrown back on the construction of the line of pressure, and this on a knowledge of the magnitude and position of the horizontal thrust H.

(*g*.) The rays of the vector polygon with polar distance H are the forces F referred to in definitions (α) and (β) above. These rays have therefore to be resolved graphically along and perpendicular to the cross-sections to find S and Q.

In actual drawing work it will accordingly be needful to know the true normals to the arch. As a rule the arch must be struck with beam compasses, and the centre will not be on the working board; hence the normals must be obtained before this centre is removed. The circle thus struck is not the arch upon which the substantial portion of the draughtmanship is done, but of this more later. If the arch be elliptic or parabolic,* provision must be made for drawing the normals at the 20 to 40 working cross-sections with considerable exactitude, otherwise it is easy to reach stresses deviating very seriously from the correct values.

(3.) *Fundamental Formulæ.*

Let ρ be the strained radius of curvature of the arch at any point, ρ_0 its unstrained value. Let A be the area of any cross-section and k its swing radius about the central axis, E the stretch-modulus, so that EAk^2 is the flexural rigidity at the given point of the arch, then with our assumptions we have the following formulæ :—

$$M=EAk^2\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right),\quad Q=-EAs_0 \quad \text{(i.)},$$

$$T=Es_0+Ey\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right) \quad \text{(ii.)},$$

$$=-\frac{Q}{A}+\frac{My}{Ak^2} \quad \text{(iii.)},$$

where s_0 is the stretch in the central line, at the given cross-section and T is the tensile stress at points of the cross-section distant y from the central axis.† Now, if h_1 and h_2 be the distances of the extreme "fibres" at the cross-section from the central axis, and T_1 and T_2 the corresponding stresses in extrados and intrados, we have

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* As long as the rise is not more than $\frac{1}{10}$ the span, the circular, parabolic, elliptic and catenary arches almost coincide, and without great vertical exaggeration no distinction can be made in drawing these curves. This supposes the curvatures of the circle, the parabola at the vertex, the catenary at the vertex, and the ellipse at the apse to be equal. If in addition we take the vertical axis of the ellipse $6c$ and its horizontal axis $2\sqrt{3}c$, where c is the radius of curvature of the catenary at the vertex; this ellipse for all practical purposes may replace the catenary on the drawing-board, and this even for catenary arches which it would be impossible to describe as "flat."

† For proof of these formulæ, see 'Drapers' Research Memoirs,' Technical Series I., p. 5, *et seq.*

$$\left.\begin{aligned}T_1&=-Q/A+Mh_1/(Ak^2)\\ T_2&=-Q/A+Mh_2/(Ak^2)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(iv.)},$$

determining the maximum stresses as soon as the curves of bending moment and total thrust on the cross-section have been constructed.

These formulæ are still for practical purposes true, if there be a slight but continuous change in the cross-section.

Let us now consider the deflections of the arch. Take the origin at the end A, and let the horizontal and vertical co-ordinates of a point P on the central line be U, V after, and U_0, V_0 before strain.

Let $u=U-V_0$, $v=V-V_0$ be the horizontal and vertical shifts. Let P′ be an adjacent point to P, and its corresponding co-ordinates be marked by dashed letters. Let ψ and ψ' be the inclination of the tangents at P and P′ to the horizontal after strain, ψ_0 and ψ_0' their values before strain. σ and σ_0 are the arcs PP′ after and before strain, so that $\sigma=\sigma_0(1+s_0)$. w and w_0 are the values of ψ and ψ_0 at A, and $\chi=\psi-\psi'$, $\chi_0=\psi_0-\psi_0'$. We have at once, if Σ denote summation from A to P,

$$\psi=w-\Sigma(\chi), \quad \psi_0=w_0-\Sigma(\chi_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(v.)},$$

$$1/\rho=\chi/\sigma, \qquad 1/\rho_0=\chi_0/\sigma_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vi.)}.$$

Hence

$$\begin{aligned}\psi-\psi_0&=w-w_0-\Sigma\left(\frac{\sigma}{\rho}-\frac{\sigma_0}{\rho_0}\right)\\ &=w-w_0-\Sigma\left\{\sigma_0\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)\right\}-\Sigma\left(\frac{\sigma_0 s_0}{\rho_0}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(vii.)},\end{aligned}$$

remembering that s_0 and $1/\rho-1/\rho_0$ are small quantities whose products may be neglected. Thus by (i.) we find

$$\psi-\psi_0=w-w_0-\Sigma\left(\frac{M\sigma_0}{EAk^2}\right)+\Sigma\left(\frac{Q\sigma_0}{EA\rho_0}\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(viii.)}.$$

Further

$$U=\Sigma(\sigma\cos\psi), \quad V=\Sigma(\sigma\sin\psi).$$

Hence

$$u=U-U_0=\Sigma(\sigma\cos\psi-\sigma_0\cos\psi_0),$$

$$v=V-V_0=\Sigma(\sigma\sin\psi-\sigma_0\sin\psi_0).$$

Let us write $b_0=\sigma_0\cos\psi_0$, $c_0=\sigma_0\sin\psi_0$, so that b_0 and c_0 are the horizontal and vertical projections of σ_0. Then we have, remembering that we may neglect the product of small quantities,

$$\left.\begin{aligned}u&=-\Sigma\{c_0(\psi-\psi_0)\}+\Sigma(b_0 s_0)\\ v&=\Sigma\{b_0(\psi-\psi_0)\}+\Sigma(c_0 s_0)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(ix.)}.$$

Now let $\epsilon = \psi - \psi_0$, as given by (viii.), and ϵ' be the value of ϵ at P′, then if η be $\epsilon' - \epsilon$, we shall have

$$\eta = -(\mathrm{M}\sigma_0)/(\mathrm{EA}h^2) + (\mathrm{Q}\sigma_0)/(\mathrm{EA}\rho_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{x.}),$$

where M and Q must be given their mean values for the element σ_0, or, if σ_0 be extremely small, their values for the mid-point of σ_0.

Now if ϵ be plotted to V_0, we have clearly, from equivalence of areas,

$$\Sigma\{c_0(\psi - \psi_0)\} = \epsilon \mathrm{V}_0 - \Sigma(\mathrm{V}_0 \eta)\,;$$

and, if ϵ be plotted to U_0,

$$\Sigma\{b_0(\psi - \psi_0)\} = \epsilon \mathrm{U}_0 - \Sigma(\mathrm{U}_0 \eta).$$

Substituting in (ix.) above and using the values found for ϵ and η in (viii.) and (x.), we find

$$\begin{aligned} u = -(w - w_0)\mathrm{V}_0 - \Sigma\left(\frac{\mathrm{MV}_0\sigma_0}{\mathrm{EA}k^2}\right) + \mathrm{V}_0\Sigma\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right) \\ + \Sigma\left(\frac{\mathrm{QV}_0\sigma_0}{\mathrm{EA}\rho_0}\right) - \mathrm{V}_0\Sigma\left(\frac{\mathrm{Q}\sigma_0}{\mathrm{EA}\rho_0}\right) - \Sigma\left(\frac{\mathrm{Q}b_0}{\mathrm{EA}}\right) \quad . \quad . \quad . \quad (\text{xi.}), \end{aligned}$$

$$\begin{aligned} v = (w - w_0)\mathrm{U}_0 + \Sigma\left(\frac{\mathrm{MU}_0\sigma_0}{\mathrm{EA}k^2}\right) - \mathrm{U}_0\Sigma\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right) \\ - \Sigma\left(\frac{\mathrm{QU}_0\sigma_0}{\mathrm{EA}\rho_0}\right) + \mathrm{U}_0\Sigma\left(\frac{\mathrm{Q}\sigma_0}{\mathrm{EA}\rho_0}\right) - \Sigma\left(\frac{\mathrm{Q}c_0}{\mathrm{EA}}\right). \quad . \quad . \quad . \quad (\text{xii.}). \end{aligned}$$

It must be remembered that in these formulæ, (xi.) and (xii.), U_0 and V_0 inside the summations Σ take every value in succession up to P; outside the summations they are the unstrained co-ordinates of the point P, the shifts u and v of which are required. (viii.), (xi.), and (xii.) contain the complete general solution.

A remark must here be made. The terms in (xi.) and (xii.) involving Q are, for all practical arches, small as compared with those in M, *i.e.*, the bending moment is far more influential in producing stress and shift than the thrust. Accordingly, by the few writers on the graphical treatment of this subject, they are put zero without further ado. Now we consider this undesirable. These terms are indeed small, but we can easily obtain a very close approximation to their values, and then we can neglect them or not, as we please. We shall then find that in certain cases they provide the substantive part of the deflections, and to neglect them is inadmissible.

Now, in "flat" arches of modern practice it will be found that the curve of total thrust on the cross-section is very nearly a straight line, that is, Q only differs from H by small quantities. Hence in the small or Q terms of (xi.) and (xii.) it is sufficiently approximate, as we neglect the squares of small quantities, to put Q = H,* and bring it outside the summation. This leaves us with the following expressions to be found :—

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* It might at first sight seem more reasonable to take $\mathrm{Q} = \mathrm{H}\cos\psi_0$, but although H contributes the bulk of the thrust on the cross-section, the effect of the vertical load is such that Q is actually rather larger than H at the terminals of the arch.

$$\Sigma\{(\mathrm{V}_0\sigma_0)/(\mathrm{EA}\rho_0)\},\ \Sigma\{(\mathrm{U}_0\sigma_0)/(\mathrm{EA}\rho_0)\},\ \Sigma\{\sigma_0/(\mathrm{EA}\rho_0)\},\ \Sigma\{b_0/(\mathrm{EA})\},\ \Sigma\{c_0/(\mathrm{EA})\}.$$

Now these involve no unknown quantities, and can be calculated as soon as the dimensions of the arch are known. Hence they provide no difficulty beyond labour of calculation, even if we treat ρ_0 and A as varying from section to section. If the curvature and cross-section of the arch be uniform, the determination of these terms is very easy ; and even if both vary, it is sufficient in these small terms to give A and ρ_0 their mean values, $\bar{\mathrm{A}}$ and $\bar{\rho}_0$ say.

Let us suppose in the unstrained state l_0 = the span, r_0 = the rise, and L_0 = the length of the central line of arch from terminal to terminal. Then our experience has shown us that we get far better results by dividing the central line of the arch into equal parts than by taking equal lengths along the horizontal, as is usual in French and American works. We take $\sigma_0 = \mathrm{L}_0/n$, where n equals 20 to 40, calculate the corresponding chord and set off the equal divisions round the arch. Through the mid-points of these divisions the verticals are drawn, upon which all the quantities are reckoned.

Returning to the thrust terms, we see that we now want to find

$$\left.\begin{array}{lll} \Sigma(\mathrm{V}_0\sigma_0) = \dfrac{\mathrm{L}_0}{n}\Sigma(\mathrm{V}_0), & \Sigma(\mathrm{U}_0\sigma_0) = \dfrac{\mathrm{L}_0}{n}\Sigma(\mathrm{U}_0), & \\ \Sigma(\sigma_0) = \mathrm{L}_\mathrm{P}, & \Sigma(b_0) = \mathrm{U}_0, & \Sigma(c_0) = \mathrm{V}_0 \end{array}\right\} \quad . \quad . \quad . \quad . \quad (\text{xiii.}).$$

The last three are fully determined, the first two are simply the moments of the portion AP of the central line treated as a wire about the horizontal and vertical through A. They can be found at once by arithmetic by reading off the co-ordinates at the verticals referred to above, adding and dividing by n, or by one or other of the usual graphical processes for finding the centroids of curves. If the arch be circular, and in these small terms it is generally sufficient to consider it circular, we have by analysis, if L_P = length of central line AP,

$$\Sigma(\sigma_0\mathrm{U}_0) = \tfrac{1}{2}l_0\mathrm{L}_\mathrm{P} - \rho_0\mathrm{V}_0, \quad \Sigma(\sigma_0\mathrm{V}_0) = \rho_0\mathrm{U}_0 - (\rho_0 - r_0)\,\mathrm{L}_\mathrm{P} \quad . \quad . \quad (\text{xiii.})\ \textit{bis}.$$

If we work only with the mid-points of the n lengths of arch σ_0, then, if P be the mid-point of the m^{th} length, $\mathrm{L}_\mathrm{P} = (m - {\cdot}5)\,\mathrm{L}_0/n$, and is thus found at once.

Substituting in (viii.), (xi.), and (xii.), we deduce the following closely approximate expressions for the deflections and change of slope :—

$$\psi - \psi_0 = w - w_0 - \Sigma\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right) + \frac{\mathrm{HL}_\mathrm{P}}{\mathrm{E}\bar{\mathrm{A}}\bar{\rho}_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xiv.}),$$

$$u = -(w - w_0)\,\mathrm{V}_0 - \Sigma\left(\frac{\mathrm{MV}_0\sigma_0}{\mathrm{EA}k^2}\right) + \mathrm{V}_0\Sigma\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right) - \frac{\mathrm{HL}_\mathrm{P}}{\mathrm{E}\bar{\mathrm{A}}}\left(1 - \frac{r_0 - \mathrm{V}_0}{\bar{\rho}_0}\right) \quad . \quad (\text{xv.}),$$

$$v=(w-w_0)\,\mathrm{U}_0+\Sigma\left(\frac{\mathrm{MU}_0\sigma_0}{\mathrm{EA}k^2}\right)-\mathrm{U}_0\Sigma\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right)+\frac{\mathrm{HL}_{\mathrm{P}}}{\mathrm{E\bar{A}}}\,\frac{\mathrm{U}_0-\frac{1}{2}l_0}{\bar{\rho}_0} \quad . \quad . \quad . \quad \text{(xvi.)}.$$

We must now apply these equations to the several special cases.

(4.) *Case* I.—*The Two-pivoted Arch.*

(*a.*) *Thrust due to Load.*

Since u must be zero at the terminal B, and V_0 is clearly there zero, we have from (xv.), if $\bar{\Sigma}$ denotes a summation throughout the arch,

$$0=\bar{\Sigma}\left(\frac{\mathrm{MV}_0\sigma_0}{\mathrm{EA}k^2}\right)+\frac{\mathrm{HL}_0}{\mathrm{E\bar{A}}}\,\frac{\bar{\rho}_0-r_0}{\bar{\rho}_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xvii.)}.$$

Now let y be an ordinate of the true line of pressure measured from the span of the arch, then by EDDY's theorem (p. 6, (*f.*)).

$$\mathrm{M}=\mathrm{H}\,(\mathrm{V}_0-y).$$

But $\mathrm{H}y$ = moment of loads treated as a system upon a simply supported beam AB, and this moment must be the same, if it were determined from the intercept y_0 of a link polygon drawn from the vector polygon with any polar distance H_0. Hence

$$\mathrm{M}=\mathrm{HV}_0-\mathrm{H}_0y_0.$$

Substituting in (xvii.), we find

$$\mathrm{H}=\mathrm{H}_0\frac{\bar{\Sigma}\left(\dfrac{y_0\mathrm{V}_0\sigma_0}{\mathrm{EA}k^2}\right)}{\bar{\Sigma}\left(\dfrac{\mathrm{V}_0{}^2\sigma_0}{\mathrm{EA}k^2}\right)+\dfrac{\mathrm{L}_0}{\mathrm{E\bar{A}}}\,\dfrac{\bar{\rho}_0-r_0}{\bar{\rho}_0}} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xviii.)}.$$

This is the general expression for the thrust due to the load. Since the form of the arch is known, r_0, L_0, $\bar{\rho}_0$, with $\mathrm{EA}k^2$ and V_0 at every section are known. Since the load is known, we can choose a suitable value of H_0, draw any link polygon for the load, and find the y_0's under each σ_0 vertical. Thus the whole of the right-hand side is determinate, and accordingly H can be found. The points to be considered are: How is H to be conveniently found, and what approximations are allowable?

For most arches it is sufficient in practice to consider $\mathrm{EA}k^2$ constant and equal to its mean value $\mathrm{E\bar{A}}\bar{k}^2$. Otherwise the summations must be actually found by arithmetic. Usually it is sufficient to put, remembering $\sigma_0=\mathrm{L}_0/n$,

$$\mathrm{H}=\frac{\mathrm{H}_0\dfrac{1}{n}\bar{\Sigma}\,(y_0\mathrm{V}_0)}{\dfrac{1}{n}\bar{\Sigma}\,(\mathrm{V}_0{}^2)+\bar{k}^2\left(1-\dfrac{r_0}{\bar{\rho}_0}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xix.)}.$$

The term in $\bar{k}^2$ is omitted by LÉVY; it is small as compared with $\frac{1}{n}\overline{\Sigma}(V_0^2)$, which is of the order of the rise squared, but not always quite negligible. We can allow for it as follows: The denominator may be written

$$\frac{1}{n}\overline{\Sigma}(V_0^2)\left\{1+\bar{k}^2\left(1-\frac{r_0}{\bar{\rho}_0}\right)\Big/\frac{1}{n}\overline{\Sigma}(V_0^2)\right\}.$$

The second $\frac{1}{n}\overline{\Sigma}(V_0^2)$ may be allowed for by treating the arch as a parabola or "flat" circular arch, in either of which cases the mean of V_0^2 may be just as closely taken along the horizontal as along the arc. The mean square of the ordinates of a parabolic segment taken along the chord is easily found to be $\frac{8}{15}r_0^2$, where r_0 is the subtense. For most arches $r_0/\bar{\rho}_0$ is small.

Thus for the Pont Alexandre III. it is only ·029 and accordingly may be omitted in this small term. We then have

$$H=H_0\frac{\frac{1}{n}\overline{\Sigma}(y_0V_0)}{\frac{1}{n}\overline{\Sigma}(V_0^2)}\Bigg/\left(1+\frac{15}{8}\frac{\bar{k}^2}{r_0^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xx.)}$$

which for most purposes is ample.

There is no difficulty, however, in finding $\frac{1}{n}\overline{\Sigma}(V_0^2)$ for a circular arch by analysis. We have accurately for n indefinitely large:

$$\frac{1}{n}\overline{\Sigma}(V_0^2)=\bar{\rho}_0^2\left\{\frac{3}{2}-2\frac{r_0}{\bar{\rho}_0}+\frac{r_0^2}{\bar{\rho}_0^2}-\frac{3}{2}\frac{l_0}{L_0}\left(1-\frac{r_0}{\bar{\rho}_0}\right)\right\} \quad . \quad . \quad . \quad . \quad \text{(xxi.)}.$$

But

$$\left.\frac{l_0}{L_0}=1-\frac{1}{3}\frac{r_0}{\bar{\rho}_0}-\frac{1}{45}\left(\frac{r_0}{\bar{\rho}_0}\right)^2-\frac{1}{189}\left(\frac{r_0}{\bar{\rho}_0}\right)^3-\dots\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxii)}.$$

Hence approximately

$$\frac{1}{n}\overline{\Sigma}(V_0^2)=\frac{8}{15}r_0^2\left(1-\frac{1}{21}\frac{r_0}{\bar{\rho}_0}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxiii.)},$$

and we find

$$H=H_0\cdot\frac{\frac{1}{n}\overline{\Sigma}(y_0V_0)}{\frac{1}{n}\overline{\Sigma}(V_0^2)}\Bigg/\left\{1+\frac{15}{8}\frac{\bar{k}^2}{r_0^2}\left(1-\frac{20}{21}\frac{r_0}{\bar{\rho}_0}\right)\right\}$$

$$=H_0\cdot\frac{\frac{1}{n}\overline{\Sigma}(y_0V_0)}{\frac{1}{n}\overline{\Sigma}(V_0^2)}\Bigg/\left(1+\frac{15}{8}\frac{\bar{k}^2}{r_0^2}-\frac{25}{14}\frac{\bar{k}^2}{r_0\bar{\rho}_0}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxiv.)}.$$

The term $\frac{25}{14}\frac{\bar{k}^2}{r_0\bar{\rho}_0}$ is, however, usually of no importance.

We now turn to methods of calculating the summations.

(A.) $\frac{1}{n}\bar{\Sigma}(V_0^2)$. This can be found in four ways :

(i.) It is approximately $\frac{8}{15}r_0^2$.

(ii.) It can be found exactly from (xxi.) if the arch be circular, and even closely for most flat arches.

(iii.) If 15 to 20 divisions of the central line of half the arch be taken, and the corresponding mid-ordinates be read off, squared by a table of squares, added, and the mean taken, we get a close value of the denominator sum.

(iv.) Let horizontal lines be drawn through the mid-points of the divisions σ_0, let the corresponding ordinate V_0 be supposed to act in these horizontal lines as a system of horizontal forces, then $\bar{\Sigma}(V_0^2)$ is the moment of this system of forces about A. In other words, a simple link- and vector-polygon construction gives $\bar{\Sigma}(V_0^2)$.

For the purpose of verification at least two of these methods should be used to find $\bar{\Sigma}(V_0^2)$ in any important bit of work. It is a constant of the arch which recurs in all the formulæ.

(B.) $\frac{1}{n}\bar{\Sigma}(y_0V_0)$. This can be found in three ways :

(i.) Read off y_0 and V_0 on the bending-moment curve for the loads, and on the central line of the arch for the mid-ordinates of the divisions σ_0, multiply corresponding values together by a slide rule or CRELLE's tables, add the products and take the mean. This gives $\frac{1}{n}\bar{\Sigma}(y_0V_0)$.

For very accurate work, such as was needful in part of the illustrations of this memoir, y_0 and V_0 may be actually calculated and not taken from the drawings.

(ii.) Treat the y_0's as a system of *horizontal* forces acting at the mid-points of the arch elements, then $\bar{\Sigma}(y_0V_0)$ is the moment of this system round A. It can therefore be found graphically by the construction of link and vector polygons.

In performing this operation (as that of (A.) (iv.) above) the y_0's are taken off the bending-moment curve for the loads on a reduced scale by the proportional compasses and placed on a horizontal vector line. Any polar distance being chosen—the same as that for (A.) (iv.) to save trouble—a link polygon is constructed on the horizontal lines through the mid-points on the arch elements ; the intercept between the first and last link made on the base AB of the arch represents on a certain scale $\bar{\Sigma}(y_0V_0)$. If $\bar{\Sigma}(V_0^2)$ has been found to the same scale, the ratio of the intercepts in the two cases gives the ratio of H to H_0 with the exception of the reducing factor $1+\frac{15}{8}\bar{k}^2/r_0^2$. The equality of the arch divisions, and not of the horizontal divisions, is an essential of our method of graphically calculating these summations.

(iii.) Draw a link polygon for the ordinates of the mid-points of the arch elements treated as a system of *vertical* forces. Let the polar distance of the corresponding vector polygon be p, and the vertical intercept of the link polygon at distance x_1 from the terminal of the arch A be i. Now considering the V_0's as a system of forces upon a simple beam of the span AB of the arch, their moment about the section at x_1 would be

$$p \times i = (l_0 - x_1)\, Q_B - \sum_{x_1}^{l_0} \{V_0(\xi - x_1)\},$$

where ξ is the distance from A to the point of application of the force V_0, and Q_B is the reaction at B of the beam-support due to the loads V_0. Hence

$$Q_B = \sum_{0}^{l_0} (V_0 \xi)/l_0,$$

and therefore

$$p \times i = \left[\frac{l_0 - x_1}{l_0} \sum_{0}^{l_0} (V_0 \xi) - \sum_{x_1}^{l_0} \{V_0(\xi - x_1)\}\right],$$

or, after some re-arranging and putting $x_2 = l_0 - x_1$,

$$p \times i = \frac{1}{l_0}\left\{x_2 \sum_{0}^{x_1} (V_0 \xi) + x_1 \sum_{x_1}^{l_0} (l_0 - \xi)\, V_0\right\}$$

$$= \frac{x_1 x_2}{l_0}\left\{\sum_{0}^{x_1}\left(V_0 \frac{\xi}{x_1}\right) + \sum_{x_1}^{l_0}\left(\frac{l_0 - \xi}{x_2} V_0\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad (\text{xxv.}).$$

Now let the load on the arch consist of an isolated load W at distance x_1 from A. Then the bending-moment curve for this load would be a triangle on the base of the arch with apex at distance $z_0 = W x_1 x_2 / l_0$ from the base. Further, y_0 from 0 to x_1 would be $z_0 \xi / x_1$, and from x_1 to l_0, $z_0(l_0 - \xi)/x_2$. Hence (xxv.) gives us

$$p \times i \times W = H_0 \overline{\Sigma}(y_0 V_0).$$

Now let β be the reducing factor of the denominator of (xx.) or (xxiv.). Then we have, if H_W be the thrust due to the isolated load W,

$$H_W = W \frac{p \times i}{\beta \overline{\Sigma}(V_0^2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxvi.}).$$

Since p is a length entirely at our choice, we can take $p = \beta \overline{\Sigma}(V_0^2)/(m r_0)$, where m is a pure numerical factor convenient for drawing. We then have

$$H_W = W \frac{i}{m r_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxvii.}),$$

or, the thrust due to an isolated load W at any point of the arch is merely W altered in the ratio of i to m times the rise. Here i is the vertical intercept of the link-polygon drawn for the V_0's as a system of vertical forces on a simple beam AB, the polar distance p being the length $\beta \overline{\Sigma}(V_0^2)/(m r_0)$.

Thus, by the construction of a *single* link polygon, we can readily obtain by summation the thrust due to any system of isolated loads on the arch. This result is of special value in the case of a system of isolated travelling loads crossing the arch.

Corollary.—If we have a continuous uniform load w per foot run of the *horizontal*, and covering any given portion of it, x' to x'', from A, we see that $W = w \times \delta x$, and therefore

$$H_w = w \times \int_{x'}^{x''} i\,\delta x/(mr_0).$$

But $\int_{x'}^{x''} i\,\delta x$ is the area of the link polygon between the ordinates corresponding to x' and x'', and this can be found at once by using a planimeter. Hence, the area between any pair of ordinates of the above link polygon of the V_0's as vertical forces enables us to find the thrust of the arch due to any uniform continuous load for the corresponding portion of the horizontal.

There is no doubt that (B.) (iii.)—originally due, we believe, to LÉVY—is of very considerable service if we want to determine the thrust for a variety of loads on the arch. But as we usually content ourselves with (a) a given dead load, and (b) the most unfavourable position of the travelling load, and (B.) (i.) is in practice more exact, (B.) (iii.) is not so advantageous as it appears at first sight.

(b.) *Thrust due to Change of Temperature.*

Let us suppose the arch has full play to expand under a change of temperature of τ from setting-up temperature, then its span would increase by $\alpha\tau l_0$, where α is the coefficient of expansion of the metal. In this case σ_0 would become $\sigma_0(1+\alpha\tau)$ and L_P become $L_P(1+\alpha\tau)$. Now bring the pivot B back to its old distance l_0 from A, then we shall have the strains in the arch which produce temperature stresses. In this case $M = H_\tau \times V_0$, where H_τ is the thrust due to temperature change. Neglecting the squares of $\alpha\tau$, we have at once from (xv.)

$$-l_0\alpha\tau = -H_\tau\left\{\bar{\Sigma}\left(\frac{V_0{}^2\sigma_0}{\mathrm{EA}k^2}\right) + \frac{L_P}{E\bar{A}}\,\frac{\bar{\rho}_0 - r_0}{\bar{\rho}_0}\right\},$$

or, using the mean flexural rigidity,

$$H_\tau = \frac{E\bar{A}\bar{k}^2\alpha\tau l_0/L_0}{\frac{1}{n}\bar{\Sigma}(V_0{}^2)\left\{1 + \frac{15}{8}\frac{\bar{k}^2}{r_0{}^2} - \frac{25}{14}\frac{\bar{k}^2}{r_0\bar{\rho}_0}\right\}} \quad . \quad . \quad . \quad . \quad . \quad (\text{xxviii.}).$$

Here l_0/L_0 may be actually determined, or found approximately from (xxii.). If the temperature stresses are small, as they very often are for pivoted arches, we can put $l_0/L_0 = 1$, neglect the correcting factor in the denominator, and put

$$\frac{1}{n}\bar{\Sigma}(V_0{}^2) = \tfrac{8}{15}r_0{}^2.$$

In this case

$$H_\tau = \tfrac{15}{8}E\bar{A}\alpha\tau\bar{k}^2/r_0{}^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxix.}),$$

and is very simple.

In the case of temperature stresses the only force to be considered on the arch is this thrust H_τ between the terminal pivots. Hence it must be resolved along a perpendicular to each cross-section to get the shear S and the cross-sectional thrust Q, then, since $M = H_\tau \times V_0$, we are able to find from (iv.) the temperature stresses in intrados and extrados and plot them at once. This part of the problem presents no further difficulty.

(*c.*) *On the Stresses in Extrados and Intrados due to Load.*

These are to be found from (iv.). As soon as the true thrust H is known, we can construct the force polygon described in (*e*), p. 6, and if the normals to the arch have been correctly drawn, the rays of the vector polygon, resolved as in (*g*), will give Q and S with all the accuracy needful in practice.

It is quite another story, however, with the true line of pressure from which the bending moments M are to be found by EDDY'S theorem ((*f*), p. 6). In a modern "flat" arch H is very considerable, and it is a part of the goodness of the design that the line of pressure and the central line of the arch should almost coincide, at any rate for the dead load on the arch. If, therefore, the arch were taken when we draw the line of pressure of the undistorted dimensions necessary in the above constructions for finding the shear and thrust on a cross-section, we should be multiplying a large force by so small an intercept that it could not be accurately read. Accordingly it is needful to exaggerate the vertical dimensions of the arch and the true line of pressure to the full extent of which the drawing-board will conveniently admit, say 8 to 10 times the drawing-board's horizontal length scale. This point has been already noted by writers on the arch; but it strikes us as curious that if they have really put their constructions into practice, experience has not taught them, as it has us, that this exaggeration is far from sufficient. The line of pressure and the central line of a good arch will still be very close together and a millimetre error in their respective positions lead to most sensible differences in the stresses and the deflections. The whole success of the calculations is in our experience bound up—especially in built-in arches—with the extreme accuracy of draughtsmanship (and, of course, size of the board) employed to construct these exaggerated forms of the central line and the true line of pressure. We can hardly believe that the authors who lightly direct their readers to exaggerate the ordinates, and publish no facsimiles of actual drawings, have ever had any wide experience of graphical work on a variety of "flat" arches. The central line to drawing-board scale has been struck with the beam compasses. Is it true to $\frac{1}{10}$ millim.? If not, what will happen when it is multiplied 10 times? We shall have errors of a millimetre, which it is impossible to permit. Again, how is it to be multiplied? The proportional compasses are never an instrument of great exactitude; scaling off and multiplying by 10 hardly better. Shall we draw the true elliptic arc, which is the exaggerated form of the circle? The difficulties are great, with its centre and one axis well off the board. No instrument we are aware of draws such an elliptic arc with the needful size and accuracy. Again, consider the line of pressure: an error in the exactness of the polar distance of its vector figure is not serious as far as multiplying the vertical intercepts by H is concerned; but such an error is vital for the construction of the line of pressure itself. A slight error in this polar distance will easily make a millimetre error in the vertical position of points on the true line of pressure. Again, the load is very likely uniform,* and thus consists of a parabola or a couple or more parabolas. Even if we take 40 arch elements to work from, the resulting link polygon may deviate, owing to want of curvature, a millimetre from the true curve, and involve serious errors in the stresses and deflections. Hence our whole experience goes to show that where possible the exaggerated central line of the arch and the line of pressure should have their ordinates calculated, and these curves be struck through the plotted values of these ordinates. This has been done in the graphical work of this paper, where the load system allows of the line of pressure being represented by two parabolic arcs, which are calculable as soon as H is known.

If the exaggerated forms of the line of pressure and the central line are accurately plotted, there is no difficulty in finding the intercepts either by scale or subtracting numerical values. Then EDDY's theorem gives the bending-moment and (iv.) the extrados and intrados stresses.

(d.) *The Vertical Deflections of the Arch due to Loading.*

We must base a graphical construction on equation (xvi.). Putting $U_0=l_0$ in (xvi.), since v is then zero, we find

$$w-w_0=\bar{\Sigma}\left(\frac{M\sigma_0}{EAk^2}\right)-\frac{1}{l_0}\bar{\Sigma}\left(\frac{MU_0\sigma_0}{EAk^2}\right)-\frac{HL_0}{2E\bar{A}\bar{\rho}_0} \quad . \quad . \quad . \quad . \quad \text{(xxx.)}.$$

Hence, substituting in the value for v, we have

$$v=\Sigma\left(\frac{MU_0\sigma_0}{EAk^2}\right)-U_0\Sigma\left(\frac{M\sigma_0}{EAk^2}\right)+U_0\bar{\Sigma}\left(\frac{M\sigma_0}{EAk^2}\right)-\frac{U_0}{l_0}\bar{\Sigma}\left(\frac{MU_0\sigma_0}{EAk^2}\right)$$
$$-\frac{H}{E\bar{A}}\left\{\frac{U_0L_0}{2\bar{\rho}_0}-\frac{L_P(U_0-\frac{1}{2}l_0)}{\bar{\rho}_0}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxi.)}.$$

The first line represents the part v_1 of v due to the bending moment, the second v_2, the part due to the thrust.

We consider these separately. Let z' be the vertical intercept at any abscissa U'_0 between the central line and the line of pressure. *This ordinate is to be considered positive if the central line is above the line of pressure.*

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* To work out the line of pressure for two uniform loads of unequal magnitude on the two half arcs is probably the best practical method of determining stresses due to uniform dead load and the unfavourable position of a uniform live load.

Then $\mathrm{M}=\mathrm{H}z'$ by EDDY's theorem, and if we use the mean value of the flexural rigidity,*

$$v_1=\frac{\mathrm{H}\sigma_0}{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2}\left\{\Sigma(z'\mathrm{U}'_0)-\mathrm{U}_0\Sigma(z')+\mathrm{U}_0\bar{\Sigma}(z')-\frac{\mathrm{U}_0}{l_0}\bar{\Sigma}(z'\mathrm{U}'_0)\right\}.$$

Now consider a system of forces z' distributed along a simple beam AB equal to the span of the arch. We have, if Q'_{A} and Q'_{B} be the reactions,

$$\mathrm{Q}'_{\mathrm{A}}=\bar{\Sigma}(z')-\mathrm{Q}'_{\mathrm{B}},\quad\text{and}\quad l_0\mathrm{Q}'_{\mathrm{B}}=\bar{\Sigma}(z'\mathrm{U}'_0).$$

Next consider the moment of any z' about the point of abscissa U_0; it equals $z'(\mathrm{U}_0-\mathrm{U}'_0)$. Hence the bending moment of the system or moment of all the forces on the A side of the point

$$=\mathrm{Q}'_{\mathrm{A}}\mathrm{U}_0-\Sigma\{z'(\mathrm{U}_0-\mathrm{U}'_0)\}$$

$$=\mathrm{U}_0\bar{\Sigma}(z')-\frac{\mathrm{U}_0}{l}\bar{\Sigma}(z'\mathrm{U}'_0)+\Sigma(z'\mathrm{U}'_0)-\mathrm{U}_0\Sigma(z')$$

$$=v_1\times\mathrm{E}\bar{\mathrm{A}}\bar{k}^2/(\mathrm{H}\sigma_0),\ \text{from the above value of } v_1.$$

Hence, if with polar distance p' we draw a link polygon for the intercepts z' between the central line and the line of pressure, and the vertical intercept of this link polygon at P be i', we have

$$v_1=\frac{\mathrm{H}\sigma_0 p'i'}{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2}\quad\ldots\ldots\ldots\ldots\quad(\text{xxxii.}).$$

Here i' would be positive if all the z''s were positive, or z' is to be considered a positive load and plotted downwards if the central line be above the line of pressure, but an upward force if the reverse be true Remembering that a system of upward forces, line of pressure above central line, gives v_1 negative or a true deflection, we shall find it easy to distinguish from the elastic line of the arch what parts of the arch are rising or falling.

Clearly p' is a length, but it must be measured on the exaggerated scale of the intercepts z'. By a proper choice of the arbitrary polar distance p' we can make i' on the drawing-board to give the actual deflections of the arch, or a quarter of them, &c., as may be most convenient.

We now turn to the deflection due to the thrust. This is a comparatively small term; if we may neglect terms of the order $(\frac{1}{2}l_0/\bar{\rho}_0)^2$—and we generally can—we may put $\mathrm{L}_0=l_0$ and $\mathrm{L}_p=\mathrm{U}_0$, and we have

$$v_2=-\frac{\mathrm{H}}{\mathrm{E}\bar{\mathrm{A}}}\frac{\mathrm{U}_0(l_0-\mathrm{U}_0)}{\bar{\rho}_0}\quad\ldots\ldots\ldots\quad(\text{xxxiii.}).$$

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* We have introduced the dashed letters here and in one or two other places to distinguish U_0, V_0, &c., at P, from the values U'_0, V'_0, &c., at a point between A and P, which latter values have to be summed. This distinction is only needful when U_0 is brought inside Σ.

This is a true *deflection* and is represented by a parabola. Now we see at once that

$$U_0(l_0-U_0)=V_0\{2(\bar{\rho}_0-r_0)+V_0\},$$

or

$$V_0=\frac{U_0(l_0-U_0)}{2\bar{\rho}_0}\Big/\left(1-\frac{2r_0-V_0}{2\bar{\rho}_0}\right).$$

Hence very approximately we may put

$$v_2=-\frac{2H}{E\bar{A}}V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxiv.});$$

or this portion of the deflection may be allowed for by subtracting from the ordinates of the arch $2H/E\bar{A}$ of their value.

A point is here worth noticing. If the arch were compressed by a uniform force H, the squeeze in σ_0 would be $H\sigma_0/(E\bar{A})$. Thus, if $U_0=\Sigma(\sigma_0\sin\psi)$, there would be a direct change, $-\Sigma\left(\frac{H\sigma_0}{E\bar{A}}\sin\psi\right)$ or $-\frac{H}{E\bar{A}}V_0$, due to the compression of the central line of the arch by the thrust. This is only one-half of the actual change, or the other half is due to the indirect action of the thrust in changing the angle ψ. Thus we conclude that the total effect of the thrust on the deflections is equally divided between change of length of arch and change of its slope.

(*e*.) *On the Horizontal Shifts of the Arch due to Loading.*

Consider the vertical system of forces z' of the last article placed on a simple beam AB; then we have seen that if Q'_A and Q'_B be the terminal reactions,

$$Q'_A=\bar{\Sigma}(z')-Q'_B, \quad \text{and} \quad Q'_B=\frac{1}{l_0}\bar{\Sigma}(z'U'_0).$$

Thus Q'_A, Q'_B, and the z's form a system of vertical forces in equilibrium. Now suppose all these vertical forces rotated round their points of application until they become horizontal, retaining their respective senses. Their sum will, of course, be still zero; they will accordingly reduce to a couple. To find this couple, take the moment round AB of the system. Clearly Q'_A and Q'_B will have no moment, and we have, if V'_0 be the ordinate of z',

$$\text{Moment}=\bar{\Sigma}(z'V'_0).$$

Now assuming that we may give EAk^2 its mean value, we have, since $M=Hz'$, from (xvii.),

$$\bar{\Sigma}(z'V'_0)=-L_0\frac{\bar{k}^2}{\sigma_0\bar{\rho}_0}(\bar{\rho}_0-r_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxxv.}).$$

Thus the system reduces to a couple of moment given by (xxxv.). This moment is usually extremely small, and for most practical purposes the system might be treated as a system in equilibrium. This assumption is, however, not necessary for our purposes.

Let the moment of all forces of the rotated vertical system Q'_A, Q'_B, and the z''s, which fall between a point P of the central line of the arch and the terminal A be taken round P, where the ordinate is V_0, and let it be $p''\times i''$, where p'' is the polar distance and i'' the horizontal intercept between the first link to Q'_A and the link from the z' corresponding to P of a link polygon drawn for the forces Q'_A, Q'_B, z''s. Then, clearly

$$\begin{aligned} p''\times i'' &= \Sigma\{z'(V_0-V'_0)\}-Q'_A V_0 \\ &= V_0\Sigma(z')-\Sigma(z'V'_0)+\frac{V_0}{l}\bar{\Sigma}(z'U'_0)-V_0\bar{\Sigma}(z') \quad . \quad . \quad . \quad . \quad \text{(xxxvi.)}. \end{aligned}$$

Now return to the expression in (xv.) for the shift u and substitute the value of $w-w_0$ given in (xxx.), we find on the supposition that we may replace the flexural rigidity by its mean value

$$\begin{aligned} u= & \frac{H\sigma_0}{E\bar{A}\bar{k}^2}\left\{V_0\Sigma(z')-\Sigma(z'V'_0)+\frac{V_0}{l_0}\bar{\Sigma}(z'U'_0)-V_0\bar{\Sigma}(z')\right\} \\ & +\frac{H}{E\bar{A}}\left\{\frac{L_0V_0}{2\bar{\rho}_0}-L_P\left(1-\frac{r_0-V_0}{\bar{\rho}_0}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxvii.)}. \end{aligned}$$

Thus the horizontal shift consists of two parts:

(*a*.) A part u_1 due to the bending. This represents the bulk of the shift, and is given by

$$u_1=\frac{H}{E\bar{A}}\frac{\sigma_0 p''}{\bar{k}^2}i'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxviii.)}.$$

Just as in the case of the corresponding part of the vertical shift, we can so select the length p'' (on the exaggerated scale of the z''s) that i'' shall be the actual shift, or the actual shift on quarter scale, &c. Thus, in order to find u_1, all we have to do is to find the reactions, Q'_A and Q'_B, due to the system of z''s applied as vertical forces to a simple beam corresponding to the span of the arch. Then we rotate these forces round all in the same sense, remembering that if Q'_A be positive, *i.e.*, act vertically upwards, it must, when rotated, act *towards* A, so as to give a negative shift, that is a shift *towards* A.

(*b*.) The part of the shift u_2 due directly to the thrust is

$$u_2=\frac{H}{E\bar{A}}\left\{\frac{L_0V_0}{2\bar{\rho}_0}-L_P\left(1-\frac{r_0-V_0}{\bar{\rho}_0}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad \text{(xxxix.)}.$$

At B this is equal to

$$-\frac{HL_0}{E\bar{A}}\left(\frac{\bar{\rho}_0-r_0}{\bar{\rho}_0}\right),$$

exactly the value of the unbalanced couple in (xxxv.), when multiplied by the proper factor $\mathrm{H}\sigma_0/(\mathrm{E}\bar{\mathrm{A}}k^2)$. We thus see how u vanishes at A and B. It is sufficient for most practical work to roughly approximate to this term. Thus we may put $\mathrm{L}_0 = l_0$, $\mathrm{L}_{\mathrm{P}} = \mathrm{U}_0$, and we have the curve

$$u_2 = \frac{\mathrm{H}}{\mathrm{E}\bar{\mathrm{A}}\bar{\rho}_0}\{\tfrac{1}{2}l_0\mathrm{V}_0 - \mathrm{U}_0(\bar{\rho}_0 - r_0 + \mathrm{V}_0)\}.$$

If we give the equally approximate value $\mathrm{U}_0(l_0 - \mathrm{U}_0)/(2\rho_0)$ to V_0, we find

$$u_2 = \frac{\mathrm{H}}{\mathrm{E}\bar{\mathrm{A}}}\left\{\frac{(l_0 - \mathrm{U}_0)(\frac{1}{2}l_0 - \mathrm{U}_0)}{2\bar{\rho}_0^2} - \frac{\bar{\rho}_0 - r_0}{\bar{\rho}_0}\right\}\mathrm{U}_0 \quad \text{. . . . . . (xl.).}$$

This is a cubic curve from which the value of u_2 may be plotted. The maximum value of the first term in the curled bracket is $\frac{1}{4}l_0^2/\bar{\rho}_0^2 = 2r_0/\bar{\rho}_0$. Hence, since $r_0/\bar{\rho}_0$ is usually a very negligible quantity in the thrust terms, we can take very closely in practice

$$u_2 = -\frac{\mathrm{H}}{\mathrm{E}\bar{\mathrm{A}}}\mathrm{U}_0 \quad \text{. . . . . (xli.),}$$

or the shifts are simply proportional to the ordinates. This, remembering the "couple" which gives the zero effect at the terminal B, is precisely the effect which would be produced on the horizontal shifts by a uniform squeeze $\mathrm{H}/\mathrm{E}\bar{\mathrm{A}}$, due to a constant thrust H on the cross-section. There is in this case no secondary effect due to the change in slope produced by the uniform compression.

Practically it does not seem a matter of very great importance to calculate the horizontal shifts of modern "flat" arches. The vertical shift is the important factor in change of shape, and in a rough appreciation of oscillatory changes. It must, however, be noted that this shift has been calculated by French engineers in the case of the Pont Alexandre III.

(5.) *Case* II.—*On the Three-pivoted Arch, or Arch with Pin-joints at the Crown and Terminals.*

(*a.*) *Thrust due to Load.*

The determination of the thrust of the arch in this case is extremely easy. The bending moment must vanish at the crown-pin. Accordingly, if any link polygon be drawn for the system of loads on the arch, with polar distance H_0, and so placed as to pass through the pins at A and B, then j being the vertical intercept of this link polygon on the central vertical of the arch, we must have, if H be the true thrust,

$$\mathrm{H} \times r_0 = \mathrm{H}_0 \times j.$$

Thus

$$\mathrm{H} = \mathrm{H}_0 j/r_0 \quad \text{. . . . . (xlii.).}$$

A link polygon drawn with H as its polar distance will pass through the three pins and give the true line of pressure.

All we have said with regard to the need for most careful construction of the exaggerated central line and line of pressure in the case of the two-pivoted arch (see pp. 16–17) holds even with greater emphasis for the three-pinned arch. Wherever it is possible in such a case, with the line of pressure extremely close to the central line, the exaggerated ordinates of both should be actually calculated and the curves plotted from the calculated ordinates.

The line of pressure being drawn, the total thrust on the cross-section, the total shear on the cross-section, the bending moment, and the stresses can be found in precisely the same manner as for the two-pivoted arch, and their discussion at length is unnecessary.

(*b*.) *Thrust due to Change of Temperature.*

It is usually assumed that no stresses arise in a three-pinned arch owing to change of temperature. As a matter of fact no *strains* arise, but secondary changes in the stresses do arise from change of shape in the arch, and therefore may be considered due to change of temperature. For example, if we suppose the distribution of load to be unchanged by temperature, the H_0j of (xlii.) will be unchanged. Accordingly we have

$$H_\tau - H = -\frac{H}{r_0}(r_t - r_0) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xliii.}).$$

Now the span of the arch remaining constant, the chord AC from terminal to crown becomes $AC(1+\alpha\tau)$ for τ degrees rise of temperature. Hence, by similar triangles,

$$r_t - r_0 : AC(\alpha\tau) :: \rho_0 : \tfrac{1}{2}AC,$$

which gives us $r_t - r_0 = 2\rho_0\alpha\tau$. Hence

$$H_\tau = H\left(1 - \frac{2\rho_0}{r_0}\alpha\tau\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xliv.}).$$

Since ρ_0/r_0 may be large, there can be a sensible change in the thrust and therefore in the stresses, although $\alpha\tau$ is very small. For example, in an arch of 227·5 metres radius and 6·5 metres rise, with $\alpha = {\cdot}000{,}0125$ and $\tau = 40°$, we should have 3·5 per cent. change in the thrust, which is a quite sensible value. Supposing this to produce 3 to 4 per cent. change in the stresses, we see that the temperature changes, although sensible—*i.e.*, outside the allowable "engineering" error,—are still not of practical importance.*

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* They have been calculated by MM. Résal and Alby for the Pont Alexandre III. See the paper in the 'Annales des Ponts et Chaussées,' referred to on p. 43.

(c.) *On the Vertical Shifts due to Loading.*

Our general equations (xv.) and (xvi.) will still hold for each semi-arch. We shall write β for $w-w_0$, and drop in the remainder of the expressions the zero subscripts except for r_0, $\bar{\rho}_0$ and σ_0, using a subscript 1 to refer to the half-arch AC, and a subscript 2 to refer to BC. U_1, u_1 will be measured from A towards B, and U_2, u_2 from B towards A. We have

$$\left.\begin{aligned} u_1&=-\beta_1\mathrm{V}_1-\Sigma_1\left(\frac{\mathrm{MV}_1\sigma_0}{\mathrm{EA}k^2}\right)+\mathrm{V}_1\Sigma_1\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right)-\frac{\mathrm{HL}_{\mathrm{P}_1}}{\mathrm{EA}}\left(1-\frac{r_0-\mathrm{V}_1}{\bar{\rho}_0}\right)\\ v_1&=\beta_1\mathrm{U}_1+\Sigma_1\left(\frac{\mathrm{MU}_1\sigma_0}{\mathrm{EA}k^2}\right)-\mathrm{U}_1\Sigma_1\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right)+\frac{\mathrm{HL}_{\mathrm{P}_1}}{\mathrm{EA}}\left(\frac{\mathrm{U}_1-\frac{1}{2}l_0}{\bar{\rho}_0}\right)\end{aligned}\right\} \quad . \quad \text{(xlv.)},$$

where u_1 and v_1 are the shifts of any point P_1 between A and C, and Σ_1 denotes a summation up to P_1 from A.

Similarly, for a point P_2 between A and C,

$$\left.\begin{aligned} u_2&=-\beta_2\mathrm{V}_2-\Sigma_2\left(\frac{\mathrm{MV}_2\sigma_0}{\mathrm{EA}k^2}\right)+\mathrm{V}_2\Sigma_2\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right)-\frac{\mathrm{HL}_{\mathrm{P}_2}}{\mathrm{EA}}\left(1-\frac{r_0-\mathrm{V}_2}{\bar{\rho}_0}\right)\\ v_2&=\beta_2\mathrm{U}_2+\Sigma_2\left(\frac{\mathrm{MU}_2\sigma_0}{\mathrm{EA}k^2}\right)-\mathrm{U}_2\Sigma_2\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right)+\frac{\mathrm{HL}_{\mathrm{P}_2}}{\mathrm{EA}}\left(\frac{\mathrm{U}_2-\frac{1}{2}l_0}{\bar{\rho}_0}\right)\end{aligned}\right\} \quad . \quad \text{(xlvi.)}.$$

Now when $\mathrm{V}_1=\mathrm{V}_2=r_0$ and $\mathrm{U}_1=\mathrm{U}_2=\frac{1}{2}l_0$, we must have

$$v_1=v_2 \text{ and } u_1=-u_2.$$

Hence we find

$$\beta_2-\beta_1=\frac{2}{l_0}\bar{\Sigma}\left\{\frac{\mathrm{M}\sigma_0(\mathrm{U}_0-\frac{1}{2}l_0)}{\mathrm{EA}k^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xlvii.)},$$

$$\beta_2+\beta_1=\frac{1}{r_0}\bar{\Sigma}\left\{\frac{(r_0-\mathrm{V}_0)\,\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xlviii.)},$$

V_0 being the unstrained abscissa measured from A, and $\bar{\Sigma}$ being as before a summation throughout the entire arch. Let the values of β_1 found from (xlvii.) and (xlviii.) be substituted in (xlv.) to find first v_1. Then after re-arranging we find, again introducing dashed letters to distinguish running co-ordinates from their value at P,

$$\begin{aligned} v_1=&\frac{\mathrm{H}\sigma_0}{\mathrm{EA}k^2}\left[\mathrm{U}_1\left\{\bar{\Sigma}(z')-\frac{1}{l_0}\bar{\Sigma}(z'\mathrm{U}'_0)\right\}-\Sigma_1\{(\mathrm{U}_1-\mathrm{U}'_1)z'\}\right]\\ &-\frac{\mathrm{H}}{\mathrm{EA}}\frac{\mathrm{U}_0(\mathrm{U}_0-l_0)}{\rho_0}\\ &-\frac{\mathrm{U}_1}{2r_0}\frac{\mathrm{H}\sigma_0}{\mathrm{EA}k^2}\left\{\bar{\Sigma}(\mathrm{V}'_0z')+n\bar{k}^2\left(1-\frac{r_0}{\bar{\rho}_0}\right)\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xlix.)}.\end{aligned}$$

Comparing this with pp. 17 and 18, we see that we may write with sufficient closeness

$$v_1 = \frac{H\sigma_0 p'i'}{EA\bar{k}^2} - \frac{2H}{EA}V_0$$
$$- \frac{U_1}{2r_0}\frac{HL_0}{EA\bar{k}^2}\left[\frac{1}{n}\bar{\Sigma}(V_0^2)\left\{1+\tfrac{15}{8}\frac{\bar{k}^2}{r_0^2}\right\} - \frac{1}{n}\bar{\Sigma}(V_0 y_0)\right] \quad . \quad . \quad . \quad . \quad . \quad \text{(l.)}.$$

The first two terms are identical with those for the case of a doubly pivoted arch. Namely, we treat the intercepts z' between the central line and the true line of pressure, which now goes through the three pivots, as vertical forces on a beam, determining their reactions Q'_A and Q'_B, then if $p'i'$ be the bending moment of this system about any point of the arch, we have the first term in v_1, the sign being determined as on p. 18. The second term is a true deflection always proportional to the ordinate of the arch. The third term is a linear term, and clearly it depends upon the quantities which we have already learnt how to determine, *i.e.*, $\frac{1}{n}\bar{\Sigma}(V_0^2)$ and $\frac{1}{n}\bar{\Sigma}(V_0 y_0)$. For example, let the thrust H_2 of a doubly pivoted arch of the same dimensions be found from the true link polygon for this three-pivoted arch. Then by (xx.)

$$H_2 = H\frac{\frac{1}{n}\bar{\Sigma}(V_0 y_0)}{\frac{1}{n}\bar{\Sigma}(V_0^2)} \bigg/ \left(1+\tfrac{15}{8}\frac{\bar{k}^2}{r_0^2}\right).$$

Hence the third term of v_1 is equal to

$$-\frac{U_1}{2r_0}\frac{HL_0}{EA\bar{k}^2}\left(1-\frac{H_2}{H}\right)\frac{1}{n}\bar{\Sigma}(V_0^2)\left\{1+\tfrac{15}{8}\frac{\bar{k}^2}{r_0^2}\right\}.$$

But, with sufficient accuracy, we may put $\frac{1}{n}\bar{\Sigma}(V_0^2) = \frac{8}{15}r_0^2$ and neglect the term $\bar{k}^2/r_0^2$. We then have the expression

$$-U_1\frac{H-H_2}{EA}\frac{4}{15}\frac{r_0 l_0}{\bar{k}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(li.)}.$$

The term $(H-H_2)/(EA)$ is usually extremely small, although the whole term may be sensible owing to $r_0/\bar{k}$ and $l_0/\bar{k}$ being large fractions.

Clearly, for the other half-arch we shall have the corresponding term

$$-U_2\frac{H-H_2}{EA}\frac{4}{15}\frac{r_0 l_0}{\bar{k}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(li.) } \textit{bis}.$$

Hence the rule is: Deal with the intercepts between the central line and the true line of pressure exactly as in the case of a two-pivoted arch. Then set down from D, the middle point of the span AB of the arch, a length DD′ equal to

$$-\tfrac{2}{15}r_0\left(\frac{l_0}{\bar{k}}\right)^2\frac{\mathrm{H}-\mathrm{H}_2}{\mathrm{EA}}.$$

Join D′ to A and B. Then the ordinates of this triangle are to be added on to the corresponding deflections.

This corrective term of the three-pinned arch may be found directly from (xlix.), and this is distinctly the shorter process, if we have to determine the horizontal shifts. Neglecting the term in curled brackets involving $\bar{k}^2$, we have to find

$$-\frac{\mathrm{U}_1}{2r_0}\frac{\mathrm{H}\sigma_0}{\mathrm{EA}\bar{k}^2}\bar{\Sigma}(\mathrm{V}'_0z').$$

Now rotate the system of z' vertical forces and the corresponding reactions Q'_A and Q'_B round their points of application until they are horizontal, Q'_A acting towards A, if it were before rotation an upward force.

Then $\bar{\Sigma}(\mathrm{V}'_0z')$ is the moment of this system of forces about $\mathrm{AB}=p'\times j'_0$, say, if j'_0 be the intercept of AB on a link polygon for this system.

Then the above expression equals

$$-\frac{\mathrm{U}_1}{2r_0}\frac{\mathrm{H}\sigma_0 p'}{\mathrm{EA}\bar{k}^2}j'_0,$$

and we may read

$$v_1=\frac{\mathrm{H}\sigma_0 p'}{\mathrm{EA}\bar{k}^2}\left(i'-j'_0\frac{\mathrm{U}_1}{2r_0}\right)-\frac{2\mathrm{H}}{\mathrm{EA}}\mathrm{V}_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lii.}),$$

a convenient form.

(*d.*) *On the Horizontal Shifts due to Loading.*

By an exactly similar process of substituting β_1 in u_1 of (xlv.) and re-arranging the terms we find

$$\begin{aligned}u_1=&\frac{\mathrm{H}\sigma_0}{\mathrm{EA}\bar{k}^2}\left[\left\{-\bar{\Sigma}(z')+\frac{1}{l_0}\bar{\Sigma}(\mathrm{U}'_0z')\right\}\mathrm{V}_1-\Sigma_1\{(\mathrm{V}_1-\mathrm{V}'_1)z'\}\right]\\&+\frac{\mathrm{H}}{\mathrm{EA}}\left\{\frac{\mathrm{L}_0\mathrm{V}_1}{2\bar{\rho}_0}-\mathrm{L}_{\mathrm{P}_1}\left(1-\frac{r_0-\mathrm{V}_0}{\bar{\rho}_0}\right)\right\}\\&+\frac{\mathrm{V}_0}{2r_0}\frac{\mathrm{H}\sigma_0}{\mathrm{EA}\bar{k}^2}\left\{\bar{\Sigma}(\mathrm{V}'_0z')+nk^2\left(1-\frac{r_0}{\rho_0}\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{liii.}).\end{aligned}$$

The first two lines of u_1 are identical in form with those for the horizontal shift of a two-pinned arch, and thus u_1 may be written with sufficient approximation

$$u_1=\frac{\mathrm{H}}{\mathrm{EA}}\frac{p'\sigma_0}{\bar{k}^2}j'-\frac{\mathrm{H}}{\mathrm{EA}}\mathrm{U}_1+\frac{\mathrm{V}_0}{2r_0}\frac{\mathrm{H}\sigma_0 p'}{\mathrm{EA}\bar{k}^2}j'_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{liv.}),$$

or, again,

$$u_1=\frac{\mathrm{H}\sigma_0 p'}{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2}j'-\frac{\mathrm{H}}{\mathrm{E}\bar{\mathrm{A}}}\mathrm{U}_1+\mathrm{V}_0\frac{\mathrm{H}-\mathrm{H}_2}{\mathrm{E}\bar{\mathrm{A}}}\tfrac{4}{15}\frac{r_0 l_0}{\bar{k}^2} \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{lv.}).$$

The value of u_2 is precisely the same except that we must read U_2 for U_1 and measure u_2 from B instead of A.

Accordingly we see that to find the horizontal shifts in the case of a three-pinned arch we must draw the true line of pressure through the three pins and place the intercepts between the central line and this line of pressure on a simple beam as vertical forces; determining the reactions due to this system, we rotate it round into the horizontal, and determine by a link polygon, with the polar distance of the vector polygon $=p'$, the intercept j' of this link polygon on the horizontal through any given point of the arch. This determines the first term. The second term is found at once by reducing the abscissæ of the arch in the ratio of H to $\mathrm{E}\bar{\mathrm{A}}$, *i.e.*, its values flow from drawing a triangle. Lastly, the third term flows from reducing in a given manner the ordinates of the arch, or by drawing a straight line making a small slope, *i.e.*, $\frac{\mathrm{H}-\mathrm{H}_2}{\mathrm{E}\bar{\mathrm{A}}}\tfrac{4}{15}\frac{r_0 l_0}{\bar{k}^2}$, with the vertical through the terminal A of the arch. Great care must be used in applying the right sign to all these component parts.

(6.) *Case* III.—*On the Doubly Built-in Arch.*

The treatment of this type is more complex than that of all the other cases, and in the actual drawings verification of results should be made at every possible stage.

(*a.*) *Thrust due to Load.*

Since both terminals are built in, the direction of the arch at both terminals remains unchanged. Hence $w=w_0$ and $\psi=\psi_0$ when we take the value at B. Thus our fundamental equations (xiv.), (xvi.) and (xv.) lead us at once to

$$\left.\begin{aligned}&\bar{\Sigma}\left(\frac{\mathrm{M}\sigma_0}{\mathrm{EA}k^2}\right)-\frac{\mathrm{HL}_0}{\mathrm{EA}\bar{\rho}_0}=0,\\ &\bar{\Sigma}\left(\frac{\mathrm{MU}_0\sigma_0}{\mathrm{EA}k^2}\right)-\frac{\mathrm{HL}_0}{\mathrm{EA}\bar{\rho}_0}\tfrac{1}{2}l_0=0,\\ &\bar{\Sigma}\left(\frac{\mathrm{MV}_0\sigma_0}{\mathrm{EA}k^2}\right)+\frac{\mathrm{HL}_0}{\mathrm{EA}\bar{\rho}_0}(\bar{\rho}_0-r_0)=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{lvi.}).$$

Whence, giving the flexural rigidity its mean value,

$$\left.\begin{aligned}&\frac{1}{n}\bar{\Sigma}(\mathrm{M})-\mathrm{H}\bar{k}^2/\bar{\rho}_0=0,\\ &\frac{1}{n}\bar{\Sigma}(\mathrm{MU}_0)-\mathrm{H}\bar{k}^2\tfrac{1}{2}l_0/\rho_0=0,\\ &\frac{1}{n}\bar{\Sigma}(\mathrm{MV}_0)+\mathrm{H}\bar{k}^2(\bar{\rho}_0-r_0)/\bar{\rho}_0=0\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{lvii.}).$$

If we make $\rho_0 = \infty$, or take a simple built-in beam, we should have the third relation identically satisfied, and the first two would reduce to

$$\bar{\Sigma}(\mathrm{M})=0, \qquad \bar{\Sigma}(\mathrm{MU}_0)=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lviii.}),$$

the usual conditions by which we construct from the bending-moment curve of a terminally supported beam the closing link, which gives us the new base for the bending-moment curve of a doubly built-in girder of the same span and loading.

Problem.—Let there be a number of verticals $y_1, y_2, y_3 \ldots y_n$ set up at distances $x_1, x_2, x_3 \ldots x_n$ from the end A of a line AB. It is required to draw a line A′B′ across these verticals so as to cut off algebraically lengths $\eta_1, \eta_2, \eta_3 \ldots \eta_n$ from their tops, so that

$$\bar{\Sigma}(\eta)=0 \quad \text{and} \quad \bar{\Sigma}(\eta x)=0.$$

These are clearly the necessary and sufficient conditions that the η's should in magnitude and position form a system of forces in equilibrium. So that the problem reduces to this: How must we cut a system of vertical forces by a straight line so that the pieces algebraically cut off their ends form a system in equilibrium?

Let the verticals through A and B meet A′B′ in A′ and B′, so that $\mathrm{AA}'=h_a$, $\mathrm{BB}'=h_b$. Join A′B. Then if $x'_r=l_0-x_r$, l_0 being the length of AB, A′B cuts off from AB a series of lengths ζ'_r, such that

$$\zeta'_r=h_a x'_r/l_0,$$

and A′B′ intercepts a series of lengths from A′B, ζ_r, such that

$$\zeta_r=h_b x_r/l_0.$$

Clearly

$$\eta_r=y_r-\zeta_r-\zeta'_r,$$

and it follows from what has been said above that the y_r system must be in equilibrium with the sum of the two systems ζ_r and ζ'_r. Let Z and Z′ be the resultants of the ζ_r and ζ'_r systems, and let them act at distance X and X′ from A and B respectively.

Then we see that

$$\mathrm{Z}=\Sigma(\zeta_r), \quad \mathrm{Z}\times\mathrm{X}=\bar{\Sigma}(x_r\zeta_r).$$

But substituting for ζ_r, we find

$$\mathrm{Z}=\frac{h_b}{l_0}\bar{\Sigma}(x_r), \quad \mathrm{Z}\times\mathrm{X}=\frac{h_b}{l_0}\bar{\Sigma}(x_r^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lix.}).$$

Thus we have

similarly

$$\left.\begin{aligned}\mathrm{X}&=\bar{\Sigma}(x_r^2)/\bar{\Sigma}(x_r),\\ \mathrm{X}'&=\bar{\Sigma}(x'^2_r)/\bar{\Sigma}(x'_r)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lx.}).$$

Clearly X and X′ are independent of the position of A′B′. Any line A′B′ will suffice to fix the verticals in which Z and Z′ act. Hence we have the following solution of the problem. Draw any arbitrary line A′B, and find the position of the resultant of the intercepts on the given verticals between A′B and AB. This determines the vertical X′; similarly the vertical X can be found. All this is simple graphical statics. Now find the forces Z and Z′, which, acting upwards in the lines X and X, would balance the y_r system of forces. This is the easy problem of determining the support reactions of a loaded beam supported at any two points. But if Z and Z′ be known, we have

$$h_b = l_0 Z/\bar{\Sigma}(x_r), \quad h_a = l_0 Z'/\bar{\Sigma}(x'_r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxi.)}.$$

The graphical solution is thus completed.

If we wish to proceed analytically, we must first find $\bar{\Sigma}(x_r)$ and $\bar{\Sigma}(x_r^2)$, then h_a and h_b are to be found from

$$\left.\begin{array}{l} \dfrac{h_b}{l_0}\bar{\Sigma}(x_r) + \dfrac{h_a}{l_0}\{nl_0 - \bar{\Sigma}(x_r)\} = \Sigma(y_r) \\ \dfrac{h_b}{l_0}\bar{\Sigma}(x_r^2) + \dfrac{h_a}{l_0}\{l_0\bar{\Sigma}(x_r) - \bar{\Sigma}(x_r^2)\} = \Sigma(x_r y_r) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxii.)}.$$

If, as in the case of arches with which we most frequently deal, the verticals, although they have asymmetrical y_0's, yet are symmetrically placed along AB, then

$$\bar{\Sigma}(x_r) = \tfrac{1}{2} n l_0.$$

Hence

$$\left.\begin{array}{l} \tfrac{1}{2}(h_a + h_b) = \dfrac{1}{n}\bar{\Sigma}(y_r), \\ h_b - h_a = l_0 \dfrac{1}{n}\bar{\Sigma}(x'_r y_r) \Big/ \left(\dfrac{1}{n}\bar{\Sigma}(x'_r)^2\right) \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxiii.)},$$

x'_r being now measured from the centre of AB towards B.

Thus, to determine h_a and h_b we require the mean values of the y_r's, of the products of y_r's into the distance from the centre of AB, and of the squares of the latter distances. Thus the problem is analytically very simple in this case. Of course, its graphical solution is simplified, for we have X′ = X, and need only find one of the X verticals. Further, we have $h_b = 2Z/n$ and $h_a = 2Z'/n$, which determine h_a and h_b at once.

We can now return to our solution of the built-in arch. Let the bending-moment curve for the load on this arch, supposed to act on a simply supported beam AB, be constructed with a polar distance H_0, and let the ordinates of this bending-moment curve corresponding to the mid-points of the equi-divisions σ_0 of the arch be, as before, y_0. Suppose a straight line A′B′ to be drawn so as to cut off portions η_0 of these ordinates, which satisfy the relations

$$\bar{\Sigma}(\eta_0) = 0 \quad \text{and} \quad \bar{\Sigma}(\eta_0 U_0) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxiv.)},$$

where we return to our arch notation U_0 for the abscissæ x. This can be done by the preceding problem.*

Now let us suppose the true line of pressure of the arch to be constructed with true polar distance H. It will be the above bending-moment curve placed in true position on the arch and drawn to a vertical scale in which y_0 is replaced by $y_0 \times H_0/H$. Now suppose A″B″ to be the line in this true line of pressure which exactly corresponds on the altered scale to A′B′. Then the portions which A″B″ cuts off from the ordinates of the true line of pressure must be $\eta_0 \times H_0/H$. Let w_0 be the distance from the central line of the arch to A″B″ and let z' be, as before, the distance between the central line and true line of pressure at any point P′, then clearly

$$z' = w_0 - \eta_0 H_0/H.$$

Thus the bending moment $= Hz' = Hw_0 - H_0\eta$.

Substitute this in (lvii.), noting (lxiv.), we have at once

$$\frac{1}{n}\overline{\Sigma}(w_0) - \bar{k}^2/\bar{\rho}_0 = 0,$$

$$\frac{1}{n}\overline{\Sigma}(w_0 U_0) - (\bar{k}^2/\bar{\rho}_0) \times \tfrac{1}{2} l_0 = 0,$$

or

$$\left.\begin{array}{l} \overline{\Sigma}(w_0 - \bar{k}^2/\bar{\rho}_0) = 0, \\ \overline{\Sigma}\{(w_0 - \bar{k}^2/\bar{\rho}_0) U_0\} = 0 \end{array}\right\} \quad \ldots \ldots \ldots \quad \text{(lxv.)},$$

since $\overline{\Sigma}(U_0) = \frac{1}{2} n l_0$.

Now suppose the arch drawn at a vertical distance $\bar{k}^2/\bar{\rho}_0$ below its true position, and let w'_0 be the distance of the central line of this re-drawn arch from A″B″, then $w'_0 = w_0 - \bar{k}^2/\bar{\rho}_0$, and therefore

$$\overline{\Sigma}(w'_0) = 0, \qquad \overline{\Sigma}(w'_0 U_0) = 0 \quad \ldots \ldots \ldots \quad \text{(lxvi.)}.$$

But these conditions are absolutely identical in form with either (lviii.) or (lxiv.). In other words, the line A″B″ is to be drawn for the re-drawn central line of the arch (*i.e.*, the original central line shifted downwards through $\bar{k}^2/\bar{\rho}_0$) in precisely the same manner as A′B′ is drawn for the link polygon of the loads on a simply supported beam, when we alter the terminals by supposing them to be built in. We shall speak of A′B′ or A″B″ as "the reduced base line." Now it is quite easy to draw the base line of a symmetrical arch. For, since the arch is symmetrical, w'_0 is symmetrical, and therefore A″B″ must be horizontal. Hence by aid of the first equation of (lxvi.) we may write the second

$$\overline{\Sigma}\{w'_0(U_0 - \tfrac{1}{2} l_0)\} = 0,$$

---

* For a circular arch $\frac{1}{n}\overline{\Sigma}(x'^2_r)$ of (lxiii.) above $= \frac{1}{L_0}\Sigma(U_0 - \frac{1}{2}l_0)^2 \sigma_0$, and this equals $\frac{1}{2}\bar{\rho}_0^2\left\{1 - \frac{l_0}{L_0}\left(1 - \frac{r_0}{\bar{\rho}_0}\right)\right\}$; or, using (xxii.) $= \frac{2}{3} r_0 \bar{\rho}_0 \left(1 - \frac{7}{15}\frac{r_0}{\bar{\rho}_0}\right)$, nearly. This will serve for most purposes.

but this is identically satisfied, since for every equal pair of w'_0's $\mathrm{U}_0-\frac{1}{2}l_0$ has corresponding equal values of opposite sign. Thus the first condition suffices to determine A″B″. But if e be its height above AB, we have

$$\bar{\Sigma}(w'_0)=\bar{\Sigma}\left(\mathrm{V}_0-\frac{\bar{k}^2}{\bar{\rho}_0}-e\right)=0.$$

Thus

$$e=\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0)-\bar{k}^2/\bar{\rho}_0 \quad \text{. . . . . (lxvii.).}$$

Accordingly e can be found at once from the mean of the ordinates to the mid-points of the equi-divisions σ_0 of the central line.

From (xiii.) *bis* we have

$$\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0)=\bar{\rho}_0\frac{l_0}{\mathrm{L}_0}-(\bar{\rho}_0-r_0)\,;$$

hence

$$e=\bar{\rho}_0\left\{\frac{l_0}{\mathrm{L}_0}-\left(1-\frac{r_0}{\bar{\rho}_0}\right)-\bar{k}^2/\bar{\rho}_0^{\,2}\right\} \quad \text{. . . . . . . (lxviii.).}$$

Or, using (xxii.),

$$e=\bar{\rho}_0\left\{\tfrac{2}{3}\frac{r_0}{\bar{\rho}_0}-\tfrac{1}{45}\left(\frac{r_0}{\bar{\rho}_0}\right)^2-\tfrac{1}{189}\left(\frac{r_0}{\bar{\rho}_0}\right)^3-\frac{\bar{k}^2}{\bar{\rho}_0^{\,2}}\right\} \quad \text{. . . . . (lxix.).}$$

For many purposes it is amply sufficient to take $e=\frac{2}{3}r_0$, which is exactly the value it would have for a parabolic arch with divisions taken equally along the *horizontal*.

In most arches, such as arise in modern practice, $\bar{k}^2/\bar{\rho}_0^{\,2}$ in (lxix.) and $\bar{k}^2/\bar{\rho}_0$ in (lxvii.) are quite negligible.

In the modified Pont Alexandre III., dealt with in this paper, $\bar{k}^2/\bar{\rho}_0$ is less than ·0006 metre, and was represented on our drawing-board by the absolutely imperceptible ·00011″.

Let us now return to the third of equations (lvii.) and substitute the value of the bending moment

$$\mathrm{M}=\mathrm{H}w_0-\mathrm{H}_0\eta_0.$$

We have at once

$$\mathrm{H}=\mathrm{H}_0\frac{\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0\eta_0)}{\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0w_0)+\bar{k}^2\left(1-\frac{r_0}{\bar{\rho}_0}\right)} \quad \text{. . . . . (lxx.).}$$

This is the equation to find the true thrust of the arch.

As on pp. 11–12, we shall now approximate to the value of the small second term in the denominator. In the first place we have

$$\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0w_0)=\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0^{\,2})-e\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0) \quad \text{. . . . . . . (lxxi.),}$$

and hence, using (xxiii.), (lxviii.), &c., we find, neglecting terms of the order $(r_0/\bar{\rho}_0)^2$ and $\bar{k}^2/(r_0\bar{\rho}_0)$, that

$$\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0w_0)=\tfrac{4}{45}r_0^{\,2}\left(1+\tfrac{1}{21}\frac{r_0}{\bar{\rho}_0}\right) \quad \text{. . . . . . . (lxxii.).}$$

Thus we obtain

$$H = H_0 \frac{\frac{1}{n}\bar{\Sigma}(V_0\eta_0)}{\frac{1}{n}\bar{\Sigma}(V_0 w_0)} \Big/ \left(1 + \tfrac{45}{4}\frac{\bar{k}^2}{r_0{}^2} - \tfrac{165}{14}\frac{\bar{k}^2}{r_0\bar{\rho}_0}\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxiii.)}.$$

The term in $\bar{k}^2/(r_0\bar{\rho}_0)$ in the correcting factor is rarely of sufficient importance to be included; the second term in $\bar{k}^2/r_0{}^2$ may be sensible in practice.

We can now describe the stages in determining the thrust and the true line of pressure of an arch.

(α.) First, e can be found from (lxvii.) or (lxix.). Or, it can be found graphically as the centre of gravity of a circular arc. The line A″B″ is then drawn and the w_0's found.

(β.) Next a bending-moment curve is drawn for the load system of the arch, placed on a simply supported beam of length equal to the arch span AB. The verticals through the mid-points of the equi-divisions σ_0 of the arch cut off intercepts, the y_0's, from this bending-moment curve. For these y_0's we determine either by calculation (see equation (lxiii.)), which experience shows to be better, or by graphics (p. 28), the position of the reduced base line A′B′ of this bending-moment curve. This determines the η_0's, or the intercepts between A′B′ and the above bending-moment curve.

(γ.) Let A′B′ meet this bending-moment curve in the points F′F′; draw verticals through F′F′ and let them meet A″B″ in F″F″, then the link polygon for the load system on the arch which passes through F″F″ and has H, as given by (lxx.), for the polar distance of its vector polygon, is the true line of pressure.

(δ.) To carry out (γ) we require to find H, and this involves a knowledge of $\frac{1}{n}\bar{\Sigma}(V_0\eta_0)$ and $\frac{1}{n}\bar{\Sigma}(V_0 w_0)$.

We may do this graphically or arithmetically.

(i.) Graphically: Treat the η_0's as a system of forces acting at the mid-points of the σ_0's horizontally, and paying due attention to their sign, according as they fall above or below A′B′. Then $\bar{\Sigma}(V_0\eta_0)$ is the moment of this system about AB, and is at once obtained by drawing the link polygon for the system and measuring its horizontal intercept i_1 on AB. Next treat the w_0's as a system of horizontal forces acting at the same mid-points of σ_0's; then the horizontal intercept i_2 on AB of a link polygon for this system measures $\bar{\Sigma}(V_0 w_0)$.

Hence, if the vector polygons have been drawn for the two cases with the same polar distances, we have

$$H = H_0 \frac{i_1}{i_2} \Big/ \left(1 + \tfrac{45}{4}\frac{\bar{k}^2}{r_0{}^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxiv.)},$$

determining H.

By (lxiv.) $\Sigma(\eta_0) = 0$, and by (lxv.) $\bar{\Sigma}(w_0) = n\bar{k}^2/\bar{\rho}_0$, and this to the scale of drawing-board work is almost zero also.* Hence, in plotting the η_0's and w_0's in the vector polygons for the above construction, we ought to start and end at exactly the same point. This will never be the case unless the lines A′B′ and A″B″ have been drawn with the greatest accuracy. Suppose, for example, that one of them is only $\frac{1}{2}$ millim. too high or too low. Then if we use, say, 40 ordinates, some 20 of these will be $\frac{1}{2}$ millim. too long and the other 20 be $\frac{1}{2}$ millim. too short; the result will be that our vector polygon will be as much as 2 centims. from closing. An error of 1 centim. to 2 centims. is thus quite consistent with very careful draughtsmanship, but such an error cannot be overlooked, as it leads to erroneous values of the thrust, the true line of pressure, and the resulting deflection. Hence we have found it absolutely necessary in drawing office practice to determine A′B′ and A″B″ by calculation, and to use the graphical construction of these lines solely for purposes of verification.†

(ii.) Arithmetically we can calculate $\bar{\Sigma}(V_0\eta_0)$ and $\bar{\Sigma}(V_0 w_0)$. Since $V_0 = w_0 + e$, and $\bar{\Sigma}(\eta_0) = \bar{\Sigma}(w_0) = 0$, we can write H in the often convenient form

$$H = \frac{\frac{1}{n}\bar{\Sigma}(w_0\eta_0)}{\frac{1}{n}\,\bar{\Sigma}(w_0^2)} \Bigg/ \left(1 + \frac{45}{4}\frac{\bar{k}^2}{r_0^2}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxv.)},$$

and calculate $\bar{\Sigma}(w_0\eta_0)$ and $\bar{\Sigma}(w_0^2)$ by aid of CRELLE's tables and BARLOW's tables.

Clearly H and the true line of pressure being known, we can now proceed, as in the two earlier cases, to find the total shears and total thrusts on the cross-sections, and the stresses in extrados and intrados.

(b.) *To determine the Thrust due to Temperature Change.*

As before, let us suppose a change of $\tau°$ on setting up temperature, involving, if the arch were free to expand, a lengthening $\alpha\tau l_0$ of the span. We shall, therefore, suppose a shift $u_{l_0} = -\alpha\tau l_0$, and we obtain from the consideration that the bending moment is solely due to the temperature thrust H_τ

$$M_\tau = H_\tau(V_0 - e_\tau),$$

where (p. 6 (*d*)) e_τ is the distance H_τ acts from AB.

But from (xiv.) for this case

$$0 = -\bar{\Sigma}\left\{\frac{H_\tau(V_0 - e_\tau)\sigma_0}{EAk^2}\right\} + \frac{H_\tau L_0}{E\bar{A}\bar{\rho}_0}\,.$$

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* For example $= \cdot 0048''$ on the full-sized drawing of the arch illustrating this case in the present paper, and $\cdot 048''$ on the exaggerated arch.

† That writers using somewhat similar constructions to those of this paper have not insisted on this point, leads us to believe that they have not put their constructions into ordinary practice. No reproductions of working drawings having so far been published, we are unable to test their experience.

Hence, as before, using the mean flexural rigidity,

$$e_\tau = \frac{1}{n}\bar{\Sigma}(\mathrm{V}_0) - \bar{k}^2/\bar{\rho}_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lxxvi.}).$$

Thus: $e_\tau = e$ of (lxvii.), or the thrust on the arch due to temperature acts in the reduced base A″B″ of the arch. It follows that $\mathrm{M} = \mathrm{H}_\tau \times w_0$ of the previous notation, and accordingly from (xv.)

$$-\alpha\tau l_0 = -\bar{\Sigma}\left(\frac{\mathrm{H}_\tau w_0 \mathrm{V}_0 \sigma_0}{\mathrm{E}\mathrm{A}k^2}\right) - \frac{\mathrm{H}_\tau \mathrm{L}_0}{\mathrm{E}\bar{\mathrm{A}}}\left(1 - \frac{r_0}{\bar{\rho}_0}\right),$$

whence we have

$$\mathrm{H}_\tau = \frac{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2 \alpha\tau l_0/\mathrm{L}_0}{\frac{1}{n}\bar{\Sigma}(w_0 \mathrm{V}_0) + \bar{k}^2\left(1 - \frac{r_0}{\bar{\rho}_0}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lxxvii.}),$$

or we can use the form

$$\mathrm{H}_\tau = \frac{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2 \alpha\tau l_0/\mathrm{L}_0}{\frac{1}{n}\bar{\Sigma}(w_0^{\,2})\left\{1 + \frac{45}{4}\frac{\bar{k}^2}{r_0^{\,2}} - \frac{165}{14}\frac{\bar{k}^2}{r_0 \bar{\rho}_0}\right\}} \quad . \quad . \quad . \quad . \quad (\text{lxxviii.}).$$

For some purposes it is sufficiently close to put $l_0/\mathrm{L}_0 = 1$, neglect the corrective factor in curled brackets and write

$$\frac{1}{n}\bar{\Sigma}(w_0^{\,2}) = \tfrac{4}{45} r_0^{\,2}.$$

We have then

$$\mathrm{H}_\tau = \tfrac{45}{4}\mathrm{E}\bar{\mathrm{A}}\alpha\tau\bar{k}^2/r_0^{\,2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lxxix.}).$$

Comparing this with (xxix.), we see that the same rise of temperature produces six times as great a thrust in the same arch when its terminals are built in as when its terminals are pivoted. But w_0 at its maximum is only about $\frac{2}{3}$ of V_0 at its maximum. Hence the maximum temperature stresses in a built-in arch are roughly about four times as great as the maximum temperature stresses in a doubly pivoted arch; in the former case they occur at the built-in ends, in the latter at the crown.

There is no difficulty, remembering that H_τ acts in A″B″, in calculating from (iii.) the actual stresses due to change of temperature.

(*c.*) *On a Method of finding the Thrust of the Arch due to an Isolated System of Travelling Loads.*

Let the load consist first of a single load W at distances x_1 and x_2 measured along the horizontal from A and B.

Using (lxxv.) we require first $\bar{\Sigma}(w_0\eta_0)$. Now η_0 is the intercept y_0 between a triangle and its base less the intercept between some sloping line (*i.e.*, the reduced base) and the base of the triangle. This sloping line must be of the form $y=m\mathrm{U}_0+q$, where m and q are constants. Therefore

$$\bar{\Sigma}(w_0\eta_0)=\bar{\Sigma}(w_0y_0)-m\bar{\Sigma}(w_0\mathrm{U}_0)-q\bar{\Sigma}(w_0)$$
$$=\bar{\Sigma}(w_0y_0),$$

since

$$\bar{\Sigma}(w_0) \quad \text{and} \quad \bar{\Sigma}(w_0\mathrm{U}_0)$$

are both zero. Hence

$$\bar{\Sigma}(w_0\eta_0)=\sum_0^{x_1}\left(w_0z_0\frac{\mathrm{U}_0}{x_1}\right)+\sum_{x_1}^{l_0}\left(w_0z_0\frac{l_0-\mathrm{U}_0}{x_2}\right),$$

where

$$z_0=\mathrm{W}x_1x_2/l_0.$$

Then, with a little re-arranging, we find

$$\bar{\Sigma}(w_0\eta_0)=\mathrm{W}\left[x_1\left\{\bar{\Sigma}(w_0)-\frac{\bar{\Sigma}(w_0\mathrm{U}_0)}{l_0}\right\}-\sum_0^{x_1}\{w_0(x_1-\mathrm{U}_0)\}\right]$$
$$=-\mathrm{W}\sum_0^{x_1}\{w_0(x_1-\mathrm{U}_0)\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxx.)},$$

since

$$\bar{\Sigma}(w_0)=\bar{\Sigma}(w_0\mathrm{U}_0)=0.$$

But

$$-\sum_0^{x_1}\{w_0(x_1-\mathrm{U}_0)\}$$

is the moment round the load W of the w_0's between A and the load. Accordingly, let a link polygon be drawn* with polar distance p for the w_0's treated as a system of vertical forces, and let i be the intercept this polygon makes on the vertical through the point of application of W.

We have

$$\mathrm{H}=\frac{\mathrm{W}pi}{\frac{1}{n}\Sigma(w_0{}^2)\left(1+\frac{45}{4}\frac{\bar{k}^2}{r_0{}^2}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxxi.)}.$$

Hence if a link polygon be drawn for the w_0's treated as an equilibrated system of forces acting at the mid-points of the equi-divisions of the arch σ_0, we can read off from its ordinates the thrust due to any system of isolated loads, and further, as on pp. 14–15, from its areas the thrust due to uniform continuous loads over any lengths of the horizontal.

The whole result can be expressed analytically if desirable. It really amounts to calculating the product moment (product of inertia) of an arc (portion of the central line) about a certain horizontal (reduced base of arch) and a certain vertical (load line of W). For practical purposes, however, the above seems the simplest mode of stating the result.

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[*] The link polygon will be a *closed* figure, since the forces are a system in equilibrium.

(*d.*) *To determine the Vertical Shift of a Doubly Built-in Arch.*

Let v be the shift at P, for which the horizontal distance from the terminal A is U_0; let z' be the vertical intercept between the central line and the true line of pressure at the point with abscissa U'_0. Then, using the mean flexural rigidity, we have from (xvi.), since $w=w_0$,

$$v=\frac{H\sigma_0}{E\bar{A}\bar{k}^2}\{\Sigma(z'U'_0)-U_0\Sigma(z')\}+\frac{HL_P}{E\bar{A}}\frac{U_0-\frac{1}{2}l_0}{\bar{\rho}_0} \quad . \quad . \quad . \quad \text{(lxxxii.)}.$$

Now

$$z'=w_0-\frac{H_0}{H}\eta_0, \quad \text{and} \quad \bar{\Sigma}(\eta_0)=\bar{\Sigma}(\eta_0U_0)=0.$$

Hence

$$\Sigma(z')=\bar{\Sigma}(w_0)=n\bar{k}^2/\bar{\rho}_0,$$

$$\bar{\Sigma}(z'U'_0)=\bar{\Sigma}(w_0U_0)=(n\bar{k}^2/\bar{\rho}_0)\times\tfrac{1}{2}l_0.$$

Using (xiv.) we can write (lxxxii.)

$$v=\frac{H\sigma_0}{E\bar{A}\bar{k}^2}\left[\Sigma\left\{\left(z'-\frac{\bar{k}^2}{\bar{\rho}_0}\right)U'_0\right\}-U_0\Sigma\left(z'-\frac{\bar{k}^2}{\bar{\rho}_0}\right)\right]-\frac{H}{E\bar{A}}V_0;$$

or,

$$v=-\frac{H\sigma_0}{E\bar{A}\bar{k}^2}\Sigma\left\{\left(z'-\frac{\bar{k}^2}{\bar{\rho}_0}\right)(U_0-U'_0)\right\}-\frac{H}{E\bar{A}}V_0 \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxxiii.)}.$$

Now the summation term is the moment of a series of forces $z'-\frac{\bar{k}^2}{\bar{\rho}_0}$ about the point where the shift is sought. In other words, suppose the arch lowered through a distance $\bar{k}^2/\bar{\rho}_0$, and the intercepts between this lowered central line and the line of pressure treated as a system of forces on a beam AB and a link polygon drawn for them with polar distance p_0, and let the vertical intercept at P be i_0. Then we have

$$v=\frac{H\sigma_0p_0i_0}{E\bar{A}\bar{k}^2}-\frac{H}{E\bar{A}}V_0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(lxxxiv.)}.$$

The shift thus consists of two parts, the first to be found by the intercept on a link polygon, the second from the reduced ordinate of the arch.

Unfortunately this method, while very simple, suffers from a defect which must be guarded against. As a rule $\bar{k}^2/\bar{\rho}_0$ is so small that it is far less than the drawing-board error, and we naturally neglect it as compared with z', for we cannot plot it. We therefore proceed to draw a link polygon for the z' pure and simple, and subtract from the shifts so calculated the terms $HV_0/(E\bar{A})$. But this is illegitimate, because although $\bar{k}^2/\bar{\rho}_0$ is insensible on the drawing-board, when *summed* in the first term of (lxxxiii.), the result is of the same order as the last term which we have retained. In practice it will usually be sufficient to deal only with the z''s and neglect both

$\bar{k}^2/\bar{\rho}_0$ and the term $\mathrm{HV}_0/(\mathrm{E}\bar{\mathrm{A}})$. Thus we find the link polygon for the z''s treated as forces on a beam AB, with polar distance p', and if i' be the intercept, we have

$$v = \mathrm{H}\sigma_0 p' i'/(\mathrm{E}\bar{\mathrm{A}}\bar{k}^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(lxxxv.)}.$$

We now compare this value of v with the maximum value of $\mathrm{HV}_0/(\mathrm{E}\bar{\mathrm{A}})$, *i.e.*, $\mathrm{H}r_0/(\mathrm{E}\bar{\mathrm{A}})$, and determine whether the latter is practically negligible. If it is, then (lxxxv.) gives the requisite vertical shift. If it is not, then we must return to (lxxxii.), which we can write in the form

$$\begin{aligned} v &= \frac{\mathrm{H}\sigma_0 p' i'}{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2} + \frac{\mathrm{HL}_{\mathrm{P}}(\mathrm{U}_0 - \frac{1}{2}l_0)}{\mathrm{E}\bar{\mathrm{A}}\bar{\rho}_0} \\ &= \frac{\mathrm{H}\sigma_0 p'}{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2}\left(i' - i'_0\frac{\mathrm{U}}{l_0}\right) - \frac{\mathrm{HU}_0(l_0 - \mathrm{U}_0)}{\mathrm{E}\bar{\mathrm{A}}\bar{\rho}_0}, \text{ nearly.} \end{aligned}$$

Here i'_0 is the value of i' at the terminal B, where i' does not truly vanish; or, since v vanishes at B,

$$i'_0 = -\frac{n\bar{k}^2}{\bar{\rho}_0}\frac{\frac{1}{2}l_0}{p'}.$$

Remembering that $\mathrm{V}_0 = \mathrm{U}_0(l_0 - \mathrm{U}_0)/(2\rho_0)$ nearly, we have

$$v = \frac{\mathrm{H}\sigma_0 p'}{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2}\left(i' - i'_0\frac{\mathrm{U}_0}{l_0}\right) - \frac{2\mathrm{H}}{\mathrm{E}\bar{\mathrm{A}}}\mathrm{V}_0 \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(lxxxvi.)}.$$

Thus we get the following easy construction. Plot the z''s down a vector line, with due regard to sign*; since $\Sigma(z')$ is not zero, but $= n\bar{k}^2/\bar{\rho}_0$, we shall not return absolutely to the point from which we start. Draw a link polygon for this system of forces, with polar distance p', *and close it by a straight line;* this is equivalent to subtracting the pieces $i'_0\mathrm{U}_0/l_0$ from i'. Now subtract from the vertical intercepts of this link polygon the vertical ordinates of the arch altered in the ratio of 2H to $\mathrm{E}\bar{\mathrm{A}}$, and we have the vertical shifts of the arch. If due attention be paid throughout to sign, the construction presents no difficulties.

(*e.*) *To determine the Horizontal Shift of a Doubly Built-in Arch.*

Returning to equation (xv.), since $w = w_0$, we have

$$u = \frac{\mathrm{H}\sigma_0}{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2}\Sigma\{(\mathrm{V}_0 - \mathrm{V}'_0)z'\} - \frac{\mathrm{HL}_{\mathrm{P}}}{\mathrm{E}\bar{\mathrm{A}}}\left(1 - \frac{r_0 - \mathrm{V}_0}{\bar{\rho}_0}\right) \quad . \quad . \quad . \qquad \text{(lxxxvii.)}.$$

Now, treat the z's as a system of horizontal forces acting at the mid-points of the equi-divisions σ_0, and draw a link polygon for them with polar distance p'' and let the horizontal intercept be i'', then we may put closely

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* z' is to be considered as a positive load and plotted downwards, if the central line be above the line of pressure. If we start with such a series of positive z''s, v will be positive, or the arch rise at the terminal. This determines which side of the vector line the pole is to be taken.

$$u = \frac{\mathrm{H}\sigma_0 p''}{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2} i'' - \frac{\mathrm{HU}_0}{\mathrm{E}\bar{\mathrm{A}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{lxxxviii.}).$$

Here we have to subtract shifts which are the abscissæ of the arch measured from A and reduced in the ratio of H to EA from the intercepts of the link polygon multiplied by a certain factor. Since $\bar{\Sigma}(z')$ is not zero, i'' at B will not be zero, but $n(\bar{k}^2/\bar{\rho}_0)(\bar{\rho}_0 - r_0)/p''$, or nearly $(n\bar{k}^2/\bar{\rho}_0) \times \bar{\rho}_0/p''$.

In working out both v and u care must be taken that p' and p'' are lengths, and lengths which must be read in the exaggerated scale of the central line of the arch and line of pressure. By a proper choice of p' and p'' we can always arrange that i' and i'' actually give the vertical and horizontal shifts of the arch to their sizes in the actual structure, or on $\frac{1}{2}$ or $\frac{1}{4}$ size, as may be more convenient.

(7.) *Case IV.—On the Pivoted Arch which carries a Metal Tie between the Pivots.*

With the treatment given in this paper for the metal arch with built-in ends, the reader will find no serious difficulty in dealing with the case of an arch built on to its tie, which may, perhaps, be considered as a closer theoretical representation of a bow-strung girder. But when we suppose the terminals of such an arch to maintain the same direction before and after strain, we are supposing the tie to be extensible and non-flexible, which can hardly be approximately the case. We ought really to suppose the tie itself as subject to couples, and the treatment then becomes somewhat more complex. Accordingly we will consider only the case of a double-pivoted arch with a metal tie of area A′ and stretch-modulus E′.*

If H be the thrust of the arch, H will be the total pull in the tie, which accordingly will be extended $\mathrm{H}l_0/(\mathrm{E}'\mathrm{A}')$, if l_0 be the length of the tie or the span of the arch.

(*a.*) *To find the Thrust of the Tied Arch.*

Turning back to equation (xv.) we have, if we put $\mathrm{V}_0 = 0$ at B and remember $\mathrm{M} = \mathrm{HV}_0 - \mathrm{H}_0 y_0$,

$$\frac{\mathrm{H}l_0}{\mathrm{E}'\mathrm{A}'} = -\bar{\Sigma}\left\{\frac{(\mathrm{HV}_0 - \mathrm{H}_0 y_0)\sigma_0 \mathrm{V}_0}{\mathrm{EA}k^2}\right\} - \frac{\mathrm{HL}_0}{\mathrm{E}\bar{\mathrm{A}}} \frac{\bar{\rho}_0 - r_0}{\bar{\rho}_0}.$$

Hence, with mean flexural rigidity,

$$\mathrm{H} = \frac{\mathrm{H}_0 \frac{1}{n} \bar{\Sigma}(y_0 \mathrm{V}_0)}{\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0{}^2) + \bar{k}^2 \left\{\frac{\mathrm{E}\bar{\mathrm{A}}l_0}{\mathrm{E}'\mathrm{A}'\mathrm{L}_0} + 1 - \frac{r_0}{\bar{\rho}_0}\right\}} \quad . \quad . \quad . \quad . \quad . \quad (\text{lxxxix.}),$$

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* We have already referred to the large arches of the St. Pancras Station as of this type. It occurs also in bridges on some of the Austrian railways.

which shows us the nature of the modification on the thrust of the arch doubly pivoted to rigid supports as given by (xix.). Let $\gamma = \mathrm{E}\bar{\mathrm{A}}/(\mathrm{E}'\mathrm{A}')$. Then if, as usual, the tie and the arch are of the same material—so that temperature changes produce no direct strains—γ is simply the ratio of the mean area of the cross-section of the arch to that of the tie. $\bar{\mathrm{A}}$ may be larger than A′, but probably not a large multiple of it, so that the k^2 term is still small as compared with $\frac{1}{n}\Sigma(\mathrm{V}_0{}^2)$. Accordingly it is usually sufficient to put*

$$\mathrm{H} = \mathrm{H}_0 \frac{\frac{1}{n}\bar{\Sigma}(y_0\mathrm{V}_0)}{\frac{1}{n}\bar{\Sigma}(\mathrm{V}_0{}^2)} \Bigg/ \left\{1 + \tfrac{15}{8}\frac{\bar{k}^2}{r_0{}^2}(1+\gamma)\right\}. \quad . \quad . \quad . \quad . \quad . \quad \text{(xc.)}.$$

This determines the thrust in precisely the same way as (xx.) for the doubly pivoted arch, except that the corrective term is now modified by the presence of γ.

The true line of pressure—passing through the two pivots—can now be drawn, the total shear and total thrust on the cross-sections determined and the stresses in extrados and intrados calculated as before.

If E=E′, we need not, as a rule, trouble about the temperature stresses. It only remains to consider the shifts.

(*b.*) *To find the Shifts of the Tied Arch.*

Following the reasoning of pp. 17–18, we see that it is not in the least invalidated by the fact that AB alters in length. This alteration affects the value of the thrust H, which is now given by (xc.) and not (xx.). This modifies the ordinates of the true line of pressure, and therefore the value of the intercepts i' determined from the link-polygon of the z''s. But the rule is the same: Take the intercepts z' between the central line and the true line of pressure and treat them as vertical forces on a simple beam AB; construct their bending-moment curve for polar distance p', then if i' be the vertical intercept of this bending-moment curve at P, the fundamental part of the deflection at P is

$$v_1 = \frac{\mathrm{H}\sigma_0 p'}{\mathrm{E}\bar{\mathrm{A}}\bar{k}^2} i' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xci.)}.$$

From this (paying due regard to its sign, p. 18) we subtract algebraically the small corrective term, v_2, which is the ordinate of the arch, altered in the ratio of 2H to EA.

In the same way the reasoning of pp. 19–20 applies to the horizontal shifts. Equation (xxxvii.) is still true. The z''s with Q'_{A} and Q'_{B} reduce still as *horizontal* forces to a couple, but its moment is not given by (xxxv.), but by

$$\bar{\Sigma}(z'\mathrm{V}'_0) = -n\bar{k}^2\left(\gamma\frac{l_0}{\mathrm{L}_0} + 1 - \frac{r_0}{\bar{\rho}_0}\right) \quad . \quad . \quad . \quad . \quad . \quad \text{(xcii.)},$$

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* Taking $l_0/\mathrm{L}_0 = 1$.

and this will generally be closely given by

$$\bar{\Sigma}(z'V'_0) = -n\bar{k}^2\left(1+\gamma-\frac{r_0}{\bar{\rho}_0}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xciii.)}.$$

If a link polygon be drawn for the Q'_A, Q'_B, and z''s treated as a system of horizontal forces with polar distance p'', and i'' be the intercept on a horizontal through P, so that $p'' \times i''$ gives the moment of all the forces between A and P, then the fundamental part of the horizontal shift at P is

$$u_2 = \frac{H\sigma_0 p''}{E\bar{A}\bar{k}^2} i'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xciv.)},$$

and all we have to do is to subtract from it the small corrective term $u_2 = -HU_0/(E\bar{A})$.

Thus the solution of the tied arch is completed.

(8.) *Illustrations.*

In order to illustrate the work of the graphical constructions, the theory of which has been discussed in the previous part of this paper, working drawings were prepared of a modified form of the Pont Alexandre III., and this arch was discussed: (*a*) on the hypothesis that its terminals were working on rigid pivots; (*b*) that its terminals were built-in; (*c*) that it had besides pivots at the terminals a pin at the crown.

Our object, besides illustrating the graphical work of these three cases, was to consider for a bridge of these dimensions the relative advantages of what we may term the Pimlico, the St. Louis, and the Pont Alexandre III. types of construction.

As for the exact dimensions of the Pont Alexandre III., the central line is parabolic and the cross-section varies from point to point. Further, we did not find it easy to ascertain the exact amount of dead and live loads to be dealt with. Hence it seemed sufficient for our purposes to adopt the following dimensions.

We assumed the arch to be circular, and

$$r_0 = \text{rise} = 6{\cdot}5 \text{ metres},$$
$$l_0 = \text{span} = 108 \text{ metres}.$$

Accordingly $\rho_0 =$ radius of the arch $= 227{\cdot}557$ metres. The drawing-board scale adopted was 1 inch = 5 metres. The arch being divided into 40 ($=n$) equal parts, each of these is $\sigma_0 = 2{\cdot}726$ metres, or in English scale $107{\cdot}3$ inches. The length L_0 of the central line of the arch is $109{\cdot}040$ metres. $r_0/\bar{\rho}_0 = {\cdot}02863$. Thus the first three terms of (xxii.) give l_0 to five places of decimals, and the first two terms would be sufficient for all practical purposes.

We took the mean cross-section of each rib to be of **I** section; equal flanges

24″×1″ and web 30″×·5″. This gave us $\bar{A}=63$ sq. inches, $\bar{k}=14{\cdot}176$ inches, $\bar{A}\bar{k}^2=12{,}661$ in inch units, and $\bar{k}^2=200{\cdot}97$ sq. inches.

As dead load we took a uniform load of 1 ton per metre run of the horizontal right across the arch. For most disadvantageous live load we supposed one half the arch to carry an additional load of $\frac{1}{2}$ ton per metre run of the horizontal. These two loads combined give a uniform distributed load of 81 tons on one half rib and of 54 tons on the other. If the more heavily loaded semi-arch be from the terminal A, we have at once

$$R_A=74{\cdot}25 \text{ tons}, \quad R_B=60{\cdot}75 \text{ tons}.$$

If we take this combined load to deduce our stresses and deflections from, we can easily find the stresses and deflections due to live and dead load separately. For we have only to re-draw stress or deflection curves inverted right and left and add the ordinates to the corresponding uninverted values, and we obtain the stress or deflection at each point due to a load of $2\frac{1}{2}$ tons per metre run; hence if these ordinates be reduced in the ratio of 2 to 5, we obtain the stress or deflection due to a uniform load of 1 ton per metre run, *i.e.*, due to the dead load. Now subtract this curve from the original curve for stress or deflection due to $1\frac{1}{2}$ tons per metre run on the one semi-arch and 1 ton per metre run on the other, and we find the stress or deflection due to the disadvantageous distribution of live load, *i.e.*, $\frac{1}{2}$ ton per metre run on one semi-arch only. This method will be found briefer than treating separately the dead and live loads, and it has been used in the present illustrations.

Illustration A.—*Doubly-Pivoted Arch.*

The line DF in Diagram I. is the span of the arch to our scale, and the flat circular arc on DF is the arch on drawing-board scale. It was struck with the beam compasses, divided into 40 equal lengths σ_0, and the verticals at the mid-points of these lengths constructed. Before the centre was lost a number of normals to the arc were constructed, with a view to determining the total shear and thrust on the cross-sections. The vertical ordinates were exaggerated 10 times to obtain the working form of the arch DEF. This was done by actual calculation of the ordinates and plotting.

The bending-moment curve for a simple beam DF, half uniformly loaded with $1\frac{1}{2}$ tons and half with 1 ton per metre run, consists of two parabolic arcs, touching at the mid-vertical. If the mid-verticals of the 40 equi-divisions of the horizontal DF be drawn—they have been removed on the drawing, as they fall extremely close to the mid-verticals of the equi-divisions of the arc—we can plot the loads on these elements of the beam, and draw the vector or force figure entitled V.P. No. 3. With pole[*] at 10 inches = 100 tons distance from the vector line the link polygon ACB was constructed and its closing link AB drawn. The intercepts of this link polygon on the arch verticals are the y_0's of the theoretical investigation. The V_0's are the arch ordinates on the same verticals. The methods of p. 13, (A.) (iv.) and (B.) (ii.), were now used to determine $\bar{\Sigma}(V_0^2)$ and $\bar{\Sigma}(y_0 V_0)$, graphically. $\frac{1}{20}$ of the V_0's and $\frac{1}{5}$ of the y_0's were plotted as horizontal forces along the vector lines V.P. No. 2 and V.P. No. 1 respectively. In the case of the V_0's, only half the arch was needful. Then with polar distances of 10 inches, the poles at O_1 and O_2 respectively, the link polygons EH for the V_0's and HEI for the y_0's were constructed between the horizontals through the mid-points of the equi-divisions σ_0 on the exaggerated arch. The distance from the mid-point of the span to H is 9·03 inches, and HI = 20·12 inches. Hence

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[*] Not shown on the drawing.

$$\bar{\Sigma}(V_0^2) = 10 \times 18{\cdot}06 \times \beta, \quad \bar{\Sigma}(V_0 y_0) = 10 \times 20{\cdot}12 \beta',$$

where β and β' are factors to reduce our board inches to actual metres. Now 1 inch = 5 metres, but our ordinates are exaggerated 10 times and were scaled off $\frac{1}{20}$ of real size. Hence $\beta = 5 \times 5 \times \frac{1}{10} \times \frac{1}{10} \times 20 = 5$. Now, the y_0's have not been exaggerated 10 times, hence $\beta' = 5 \times 5 \times \frac{1}{10} \times 5 = 12{\cdot}5$, for the y_0's were plotted $\frac{1}{5}$ of their size. Thus we have

$$\bar{\Sigma}(V_0^2) = 903 \text{ sq. metres}, \quad \bar{\Sigma}(V_0 y_0) = 2515 \text{ sq. metres}.$$

We will now investigate the corrective factor of equation (xxiv.) for the thrust, *i.e.*,

$$1 + \tfrac{15}{8} \frac{\bar{k}^2}{r_0^2} - \tfrac{25}{14} \frac{\bar{k}^2}{r_0 \rho_0}.$$

We have $\bar{k}^2 = 200{\cdot}97$ sq. inches $= {\cdot}12965$ sq. metre. Hence the corrective term is $1 + {\cdot}00575 - {\cdot}00016 = 1{\cdot}0056$. For practical purposes the third term is not worth considering.

Substituting in (xxiv.) and remembering that $H_0 = 100$ tons, we find for the true thrust of the arch by purely graphical calculus

$$H = 277 \text{ tons}.$$

Treated analytically, there is no difficulty in calculating the actual value of y_0. We have on the right of the mid-line

$$p \times y_0 = 60{\cdot}75\,(108 - U_0) - \tfrac{1}{2}(108 - U_0)^2,$$

and on the left

$$p \times y_0 = 74{\cdot}25\,U_0 - \tfrac{3}{4}U_0^2,$$

both in metre tons. This follows from the values of R_A and R_B.

Now, p was taken = 10 inches = 100 tons, and the scale is 5 metres to the inch. Hence in inches, on right

$$y_0 = {\cdot}1215\,(108 - U_0) - {\cdot}001\,(108 - U_0)^2;$$

on left

$$y_0 = {\cdot}1485\,U_0 - {\cdot}0015\,U_0^2,$$

whence the actual values of y_0 can be found.

Calculation gave

$$\bar{\Sigma}(V_0^2)=898 \text{ sq. metres} \quad \text{and} \quad \bar{\Sigma}(V_0 y_0)=2515 \text{ sq. metres.}$$

Hence from (xxiv.) we determine the thrust

$$H=278 \text{ tons.}$$

This agrees as well as could be expected with the graphical work, and 278 tons was used for the value of H.

To get the true line of pressure we must alter the ordinates y_0 in the ratio of H_0 to H, *i.e.*, 100 to 278, but we have to multiply by 10 to exaggerate. O_3 is the pole at distance 278 tons and on the horizontal which divides the vertical vector line V.P. No. 3 in the ratio of R_A to R_B. O_3' is the pole at $\frac{1}{10}$ of the distance of O_3 to get the ordinates exaggerated 10 times. The link polygon which gives the line of pressure must then be drawn between the verticals through the mid-points of the 40 equal divisions of the horizontal. This graphical method being hardly exact enough, the calculated values of the y_0's were altered in the ratio of $10H_0$ to H, *i.e.*, 1000 to 278, and the true line of pressure, the broken line DEF, constructed. The vertical intercepts between the central line and the line of pressure, the z's of the theory, are now known. Hence EDDY'S theorem, bending moment $= Hz'$, gives the bending moments. These are plotted in metre tons in the curve ϵ. Next from V.P. No. 3 and pole O_3 at 20 sections along the arch, where the normal had been drawn at point corresponding to 20 equi-divisions along the horizontal, the total shear and total thrust on the cross-sections were determined, and plotted in the curves of shear and thrust shown in the diagram. From the bending-moment and thrust curves the stresses in extrados and intrados at 40 points along the arch were found by equation (iii.) and the curves of stress in extrados and intrados plotted.

In the next place, the temperature thrust H_τ was found from (xxviii.) on the hypothesis that the coefficient of linear expansion $\alpha = {\cdot}0000125$, and that 40° C. is the limit of change of temperature from setting-up temperature.

Taking

$$\Sigma(V_0^2)=900 \text{ sq. metres,*}$$

we find

$$H_\tau=2{\cdot}42 \text{ tons.}$$

Again, by (iii.), the temperature stresses in extrados and intrados were determined at 20 sections on the half-arch, the other half being given by symmetry. These were plotted, since they were relatively small, on five times the scale of the load stresses at the bottom of the diagram.

Lastly, we turn to the vertical deflections. Consider first the term proportional to the ordinate of the arch, *i.e.*,

$$v_2=-\frac{2H}{EA}V_0,$$

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* This value is about midway between the graphical and analytical values. Formula (xxiii.) gives the value 900·1.

and let us take E = 30,000,000 per sq. inch. Hence we have

$$v_2 = - \cdot 00066\,V_0.$$

Now the maximum value of V_0 is the rise = 650 centims., thus the maximum value of v_2 will be about ·43 centim. We have plotted the deflections $\frac{1}{4}$ true deflections of arch. Hence the maximum corrective term on v_1 to our scale is about 1 millim. We have attempted to allow for this, but it is obviously a small part of the total deflection.

Next we turn to v_1 as given by (xxxii.). We have to draw a link polygon for the intercepts z' treated as vertical forces. The z''s are plotted down the vertical vector line in V.P. No. 4. Now if p' were taken exactly $= E\bar{A}k^2/(H\sigma_0)$, then i' on drawing-board scale would give v_1. Now, the drawing-board scale = 5 metres to the inch, or is 1/(196·85) of real size. Therefore, if we took $p' = E\bar{A}k^2/(H\sigma_0 \times 196{\cdot}85)$, we should get the deflections actual size. The deflections would then be too big, so that we will take p' four times this value and get the deflections $\frac{1}{4}$ actual size. Lastly we must remember that the z''s are exaggerated 10 times, and we have used these values in plotting them. It follows that p' also must be exaggerated 10 times. Further, 5 metres to the inch is the drawing-board scale. Hence we must take

$$p' = \frac{E\bar{A}k^2}{H\sigma_0} \times \frac{10 \times 4}{196{\cdot}85} \times \tfrac{1}{5} \text{ inch} = 5{\cdot}866 \text{ inches.}$$

This is the value of p' from which the link polygon for the z''s was actually constructed. The pole being at O_4, any link polygon is drawn for the z''s between the verticals through the mid-points of the σ_0's. The closing link of this polygon between the terminal verticals is put in, and the intercepts, reduced to the base $\alpha\beta$, give the curve γ of deflection due to combined live and dead loads. By the reversing process indicated on p. 40, the curve of deflections due to the dead load alone has then been obtained.

This completes the solution as far as it seems desirable to go by way of illustration. It will be seen that our maximum compressive stress is about 9·25 tons per sq. inch, and that we have a maximum deflection of about 34 centims. These are excessive, and we should certainly need to increase the stiffness or lessen the load. At the same time, the engineers of the actual Pont Alexandre III.—and they have been confirmed by the calculation of M. JACQUIER*—found for that bridge maximum deflections under total live and dead loads which amount to almost as large a value,

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* RÉSAL et ALBY, "Construction du Pont Alexandre III." 'Annales des Ponts et Chaussées,' Part II., 1898, 4e Trimestre, pp. 59–144; Part III., 1899, 1er Trimestre, pp. 159–241. JACQUIER, "Le calcul graphique des ponts en arc à triple articulation par la méthode des lignes d'influence." 'Annales des Ponts et Chaussées,' 1903, 1er Trimestre, pp. 265–300. This is an application largely arithmetical to the Pont Alexandre III. It is, perhaps, needless to say that we personally much prefer the methods adopted in the present memoir.

i.e. about 30 centims. The extreme flexibility of pivoted bridges seems a great disadvantage in this form of construction, and some attempt to counteract it appears to have been made by a particular type of cross-bracing between the ribs in the case of the Pont Alexandre III. itself.

Illustration B.—*Three-pivoted Arch.*

The drawing work for this is represented in Diagram II. All the preliminary work is precisely the same as for the two-pivoted arch. V. P. No. 1 is the vector figure for the loads on the arch treated as a load on a beam equal in length to the span of the arch divided into 40 equal pieces. The link polygon for this was drawn between verticals placed at the mid-points of the horizontal equal pieces as in Diagram I., and then, if j be the ordinate of this link polygon at the rise of the arch and H_0 be the polar distance, we have, as in (xlii.), $H = H_0 j/r_0$, giving the true thrust. Taking $\frac{1}{10}H$ as polar distance, we construct the exaggerated line of pressure of the arch passing through the pin at the crown C. This construction, although actually made, is not shown on the drawing, as it was found necessary for purposes of accuracy to correct by actual calculation the ordinates of the so constructed line of pressure. We proceeded as follows:

Let $p =$ load per unit run on left-hand semi-arch, then

$$M = -HV_0 + R_A U_0 - \tfrac{1}{2} p U_0^2$$

for the left-hand semi-arch. But $M = 0$ at the crown, for it is pinned. Therefore putting $V_0 = r_0$, $U_0 = \frac{1}{2} l_0$,

$$H = \frac{l_0}{2r_0}(R_A - \tfrac{1}{4} p l_0).$$

Now $pl_0/z = W$, total load on left-hand semi-arch, and if W′ be total load on right-hand semi-arch, we have at once $R_A = \frac{3}{4}W + \frac{1}{4}W'$, and therefore

$$H = \frac{l_0}{8r_0}(W + W') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(xcv.)}.$$

This gives in our case $H = 280{\cdot}38$ tons, or for practical purposes 280 tons.

Now let Y_0 be the ordinate of the true line of pressure measured from the base AB, then

$$M = H(Y_0 - V_0).$$

Hence

$$Y_0 = U_0(R_A - \tfrac{1}{2} p U_0)/H,$$

or

$$Y_0 = \frac{8r_0}{l_0} U_0 \left(\frac{3W + W'}{4(W + W')} - \frac{W}{W + W'} \frac{U_0}{l_0} \right)$$

$$= r_0 \frac{U_0}{l_0} \left(\frac{6W + 2W'}{W + W'} - \frac{8W}{W + W'} \frac{U_0}{l_0} \right) \quad . \quad . \quad . \quad . \quad . \qquad \text{(xcvi.)}$$

A precisely similar expression holds for Y_0 on the right semi-arch, if we interchange W and W′ and replace U_0 by $l-U_0$, *i.e.*, the abscissa measured from B. In our particular case $W/W'=\frac{3}{2}$. The ordinates of the true line of pressure were actually calculated at 40 points and plotted, and so the two parabolic arcs AC and CB which give the true line of pressure obtained. The curve of bending moment was found by subtracting corresponding values of Y_0 and V_0 and multiplying the result by H. By aid of V.P. No. 1 with the pole at $O_1=H$, the total shear and total thrust on 21 cross-sections (equally distributed along the horizontal) at which the normals had been drawn were determined. Thus the curves of shear and thrust were found. From the curve of thrust and the curve of bending moment the stresses in extrados and intrados were calculated by aid of (iii.) at 40 sections equally distributed along the arch. Thus the solution was completed up to the deflections.

To find the deflections we must turn to (lv.) and calculate the three separate parts there given. The first and third terms are to be got in precisely the same way as for a two-pivoted arch. Since H has only changed from 278 to 280, the term $-(2H/EA)V_0$ still contributes only a millimetre to the deflection at a maximum, if we adopt the same scale of deflection $=\frac{1}{4}$ their true values. An elliptic arc of rise 1 millim. and chord 21·6 inches can only be allowed for by the most rough approximation.

The main term of v_1, or $H\sigma_0 p'i'/(E\bar{A}\bar{k}^2)$, was treated exactly as in the case of the two-pivoted arch. To obtain the deflections $\frac{1}{4}$ true size it was necessary to take $p'=5{\cdot}824$ inches. The vector polygon of the intercepts between central line and true line of pressure is given in V.P. No. 2. With O_2 at polar distance p' the link polygon has been drawn, and reduced to base DE; this is broken curve DHE. We next proceeded* to find the term $\bar{\Sigma}(V'_0z')$; this was found by arithmetic and not by graphics, as being in this case more exact. Its actual value or $p'j'_0$ was found to be 2·1147 sq. metres. Giving U_1 its maximum value of 54 metres, we find for the maximum value of this term 6·12 centims. Accordingly from the middle of the span a distance to $G=1{\cdot}53$ centims. was plotted vertically downwards and G joined to D and E. This is shown by the broken lines DG and GE. This portion of the shift v was now combined with the previous result DHE, and the total deflection curve for dead and live loads as indicated on the drawing obtained. By the process described on p. 40, the portion of the deflection due to dead load was separated from that due to both dead and live loads, and thus the whole solution completed. The maximum stress is in this case a compression of about 9·75 tons per sq. inch, and the maximum shift a rise of about 25·6 centims. RÉSAL and again JACQUIER in dealing with the actual arch appear for their permanent charge to have a thrust a little more than twice our value, and the one obtains a *central* deflection of 13·5 centims. and the other one of 15·3 centims. Our central deflection due to dead load only is about 1·8 centims., which thus corresponds to 3·6 centims. with their thrust. But the shape and flexural rigidity of their arch differ from ours. The very large shift, however, due to asymmetrical live load appears in this case as in the case of the two-pivoted arch to be the crucial difficulty.

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* See pp. 24–25.

Illustration C.—*The Doubly Built-in Arch.*

The initial stages are precisely the same as for the two- and three-pivoted arches. It is only when we come to the determination of the true line of pressure on the exaggerated scale that the real difficulties of the draughtsmanship begin. Looking at Diagram III., we see that the bending-moment curve for a simple beam of span AB under the same system of load was first constructed for the mid-verticals of 40 equal divisions taken along the horizontal—these verticals have been rubbed out not to confuse the final drawing. This is AC′B′ which is reduced to the horizontal as ACB.* The exaggerated arch was plotted from calculated ordinates, and the next stages are to find the reduced bases of arch and bending-moment curve for the 40 mid-verticals of the equi-divisions of the arch. In the first place, we find e, the height of the reduced base of the arch, from (lxvii.). This is 4·320 metres, which gives us 8·64 inches on the drawing-board. This is the line QR. In the next place, a line A′B was drawn, cutting off a series of intercepts on the mid σ_0 verticals above AB. The resultant of these intercepts was then found by means of the vector polygon V.P. No. 1, and the link polygon GG′. The first and last links of this, GE, EG′, meet in E, and hence the vertical EE′ was determined corresponding to the line at distance X from A of p. 27 ; the line X′ or FF′ was put in by symmetry. Next a link polygon was drawn for the y_0's of the bending-moment curve ACB. These are plotted in V.P. No. 2, and the link polygon is DD′ ; the first and last links of this polygon meet EE′ and FF′ in E′ and F′, and consequently E′F′ is the closing link. The parallel to E′F′ through O_2 is O_2U and cuts off U0 and 40 U, on the vector line, which would be the upward forces in EE′ and FF′ balancing the downward system of y_0's. Thus we have the Z and Z′ of (lxi.), and can find h_a and h_b, the AH and BI of the drawing. These determine the reduced base line of the bending-moment curve, *i.e.*, HI. Now an error of almost the thickness of a line in HI will upset the further work, and accordingly the actual values of AH and BI were directly found by calculation, as indicated in (lxiii.). This was fairly simple, because the values of the y_r's $(=y_0$'s) and x_r's $(=\mathrm{U}_0-\frac{1}{2}l_0)$ were already known. We have in drawing-board scale

$$h_a=2{\cdot}609 \text{ inches}, \qquad h_b=2{\cdot}241 \text{ inches};$$

these values agreed closely with the graphical values.

If the reduced base HI meets the curve ACB in JK, and the verticals through J and K meet the reduced base of the arch in J′ and K′, these latter points are the points through which the true line of pressure of the doubly built-in arch passes.

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* As usual, this curve was verified by plotting the calculated ordinates.

Our next stage was to determine the true thrust from (lxxiii.). We have first to evaluate the corrective factor

$$1+\tfrac{45}{4}\frac{\bar{k}^2}{r_0{}^2}-\tfrac{165}{14}\frac{\bar{k}^2}{r_0\bar{\rho}_0}\text{, and this equals } 1+{\cdot}03450-{\cdot}00106=1{\cdot}0344.$$

$H_0=100$ tons in our construction, and $\bar{\Sigma}(V_0\eta_0)$ and $\bar{\Sigma}(V_0w_0)$ were determined in two ways, first by arithmetic, and then by graphics. The values were in sensible agreement. Looking at Diagram III., we see that horizontals were drawn through the mid-points of the exaggerated elements σ_0, and the w_0's, *i.e.*, the ordinates of the arch, plotted as horizontal forces in V.P. No. 3. Then with polar distance = 10 inches and pole at O_3, the link polygon LM was constructed for half the arch. The distance from the mid-point of the span to M is 7·57 inches, and its double for the whole arch is 15·14 inches. Similarly treating the η_0's, or distances of ACB from its reduced base, as horizontal forces at the mid-points of the σ_0's, we have the vector polygon V.P. No. 4; using the same polar distance, we construct the link figure NLP, and the intercept NP = 8·52 inches. Now for the w_0's the values on the vector figure were $\frac{1}{4}$ the exaggerated values. Hence in metres, 5 to the inch,

$$\bar{\Sigma}(w_0V_0)=15{\cdot}14\times5\times4\times10\times5/100=151{\cdot}4 \text{ sq. metres.}$$

The η_0's were taken half size, thus

$$\bar{\Sigma}(\eta_0V_0)=8{\cdot}52\times5\times2\times10\times5/10=426 \text{ sq. metres.}$$

Hence we deduce the thrust $H=272{\cdot}02$ tons, or, for practical purposes, 27·2 tons.

The ratio of $H_0/H=3{\cdot}676$. All the ordinates η_0 were multiplied by this and plotted to the reduced base line of the arch QJ′K′R, and so the true line of pressure found.* The intercepts between the central line of the arch and the true line of pressure, multiplied by the quantity H, were determined, and thus the curve of bending moment plotted. From the vector polygon V.P. No. 5 the curves of shear and thrust were determined, and finally the stresses in extrados and intrados in the usual manner.

The deflection must now be determined from (lxxxvi.). The intercepts between the true line of pressure and the central line were plotted as vertical forces down the vector line in V.P. No 6, each intercept being doubled before plotting. 40 will not quite fall on 0 for the vector line, the interval should be $n\bar{k}^2/\bar{\rho}_0$, which equals in our case ·02286 metre, or, on the scale of the drawing-board, ·00457 inch. Now, the ordinates of the arch have been exaggerated 10 times and the intercepts plotted to double size. Hence in V.P. No. 6, 0, 40 should be 20 times ·00457 inch = ·09 inch about—say, with the drawing accuracy possible, $\frac{1}{10}$ inch. It was found to be about this on the drawing. A link polygon was then drawn between the mid σ_0 verticals, closed, and reduced to the horizontal. We have thus evaluated the term

---

* This was also verified by the graphical construction from the vector polygon V.P. No. 5, with pole at O'_3.

$$\frac{H\sigma_0 p'}{EAk^2}\left(i' - i'_0 \frac{U_0}{l_0}\right).$$

The corrective term $2HV_0/EA$ again amounts to about a millimetre at its maximum.

Since we have doubled the intercepts, we must double p', and accordingly if p' be still chosen so as to give the deflections to $\frac{1}{4}$ real size, we find we must take $p' = 11{\cdot}997$ inch. This is the value of the polar distance of V.P. No. 6. The deflection curve for the total load is $\alpha\delta\beta$. Using the process indicated on p. 40, the deflection curve for the dead load only is $\alpha\epsilon\beta$.

Lastly we have to deal with the temperature stresses. The thrust due to change of temperature is given by (lxxviii.). We find H_τ for the same total change from setting-up temperature, as in the preceding cases, equal to 13·97 tons. Hence the temperature stresses in extrados and intrados were found, and they are plotted to the same scale as the load stresses at the foot of the drawing.

The maximum deflection due to loading is 19·6 centims.

The maximum stress is in the intrados at the built-in terminal, and is 10·75 due to load and 3·22 tons per sq. inch due to change of temperature. A possibility of a compressive stress of 13·97 tons seems to place this type of arch in a lower place than the pivoted arch, but it must be noticed that it occurs only at the terminal, where it is perfectly easy to strengthen the cross-section. The maximum stress away from the terminal is only 7·54 tons in extrados due to load and 0·81 ton per square inch due to temperature change; in all, a possibility of 8·4 tons, and this is sensibly less than what we reach with the other types. This supposes that there may be a rise or fall of 40° C. from setting-up temperature. We will now pass to a general discussion of the relative advantages of the three types.

(9.) *On the Relative Advantages of Two-Pivoted and Three-Pivoted and Doubly Built-in Arches.*

The construction of the marvellously graceful Pont Alexandre III. on the three-pin system has naturally led to the impression that there are some special mechanical advantages in this form of construction. As a result of our treatment in this memoir of a bridge roughly resembling the Pont Alexandre III., we are inclined to doubt whether the three-pinned construction has any marked theoretical advantages over the other two types. Even if it had, it has not yet been shown superior to a three-pinned arch with built-in terminals, the pins being at the crown and at points between the crown and springings hereafter to be determined. Nor again has it been compared with a two-pivoted, cantilever girder arch, *i.e.*, a two-pinned arch with built-in terminals, and the pins at points between the crown and springings.

Such possible structures will no doubt be considered in due course, but on the basis of our present experience we should be inclined to say that no marked advantage is likely to be gained by any special number or arrangement of pins, or by the nature of the terminal fixings, so long as we maintain what is essentially a pure arch structure.

For comparative purposes we may consider the following points:—

(A.) *Horizontal Thrust of the Arch.*

This is of some importance in considering the construction of the supports. We have total thrust $= \mathrm{H} + \mathrm{H}_\tau$, and we find:

For two-pivoted arch, $\mathrm{H} + \mathrm{H}_\tau = 278{\cdot}0 + 2{\cdot}4 = 280{\cdot}4$ tons,
" three-pivoted arch, " $= 280{\cdot}4 + 0 = 280{\cdot}4$ "
" doubly built-in arch, " $= 272{\cdot}0 + 14 = 286{\cdot}0$ "

Thus the three-pivoted arch is not much better than the two-pivoted, both would be inferior to the doubly built-in arch were it not for the large addition due to the temperature changes. The whole advantage therefore—and it is not much, about 2 per cent. in the value of the thrust—is due to temperature change. By setting up the doubly built-in arch at about the highest summer temperature we might actually profit by its temperature thrust, but in the case, as we shall see later, we should not profit by the temperature stresses. Anyhow, no great relative advantage appears in the matter of thrust.

(B.) *Stresses due to Dead Load of* 1 *Ton per Metre Run.*

Two-pivoted arch, maximum compression $= 4{\cdot}08$ is in extrados and at crown.

Three-pivoted arch, maximum compression $= 3{\cdot}97$ is in intrados and at sections at about $\frac{1}{5}$ span.

Doubly built-in arch, maximum compression $= 4{\cdot}89$ is in intrados and at terminals.

None of the arches exhibit tension under dead load. For practical purposes it is clear that the two-pivoted and three-pivoted arches exhibit an equal maximum stress, while the doubly built-in arch has a stress about ·9 ton per sq. inch larger. This is sensible, but so far not of much practical importance. On the other hand, the maximum stresses in the built-in arch occur just at the terminals, where it is perfectly easy and artistically sound to increase slightly the cross-section; the maximum stresses in the pivoted arches occur either at the crown or about $\frac{1}{5}$ span from the terminals, where it is less easy to increase the cross-sections. If we exclude the terminal cross-sections, the maximum stress in the doubly built-in arch is about 4·15 tons, and occurs in the extrados at the crown. Thus for practical purposes we might say that all three arches have a maximum dead-load stress of 4 tons per sq. inch, and that there is no really sensible difference in efficiency between them.

(C.) *Stresses due to Combined Dead and Live Loads; the Live Load being* $\frac{1}{2}$ *Ton per Metre Run on one half the Arch.*

Two-pivoted arch, maximum compression = 9·28 tons in extrados at $\frac{3}{4}$ span from A.

Three-pivoted arch, maximum compression = 9·31 tons in intrados at $\frac{1}{4}$ span from A.

Doubly built-in arch, maximum compression = 10·75 tons in intrados at terminal A.

The terminal on the more loaded side is the one designated A.

Again we see about 1·4 tons per sq. inch advantage to the pivoted arches; the three-pivoted arch, indeed, being no better than the two-pivoted. The maximum stress, however, in the doubly built-in arch occurs just where we can most easily deal with it. If we disregard the terminal stress, the maximum stress in an arch of this type is only 7·54 tons, and occurs in the extrados at a section about a third span from the terminal A of the heavier loaded semi-arch.

All three arches exhibit some tension in the stress curves :—

Two-pivoted arch, maximum tension is ·40 ton in intrados at $\frac{1}{4}$ span from terminal A.

Three-pivoted arch, maximum tension is ·31 ton in extrados at rather less than $\frac{3}{4}$ span from A.

Doubly built-in arch, maximum tension is 1·18 tons in extrados at terminal A.

The pivoted arches are both better here than the built-in, but in no case is the tension of any practical importance, and it cannot be given any weight in the relative advantages of the three types.

We see again that the three-pivoted arch, as far as stress due to combined load is concerned, has no advantage over the two-pivoted arch. Both have a less stress than the doubly built-in arch, but if we exclude the terminals of the latter arch, which can be easily enlarged in section, then the doubly built-in arch has a maximum compressive stress almost 2 tons per sq. inch less than those of the pivoted arches, and the section of this arch, apart from the terminals, may be thus relatively smaller.

(D.) *Temperature Stresses. Change of* 40° *C. from Setting-up Temperature.*

All three arches have *secondary* temperature stresses due to change in thrust, &c., following on change in shape of arch. We shall not consider these changes, they are slight and will not be very different in value for the three types (see p. 22). If we take direct temperature stresses, we have :—

Two-pivoted arch, maximum stress is ·82 ton in the intrados at crown.
Three-pivoted arch, maximum stress is zero.
Doubly built-in arch, maximum stress is 3·22 tons in the extrados at the terminals.

Clearly here the three-pivoted arch has an advantage over the two-pivoted, and this a still greater advantage over the doubly built-in arch. If we exclude the stresses at the terminal sections, we find the doubly built-in arch has a maximum temperature stress at the crown of 1·73 tons in the intrados. Further, it would have a total maximum stress due to dead load, live load, and temperature of only about 6·73 tons occurring in the extrados about $\frac{7}{24}$ span from the terminal A; this is apart from terminal stress. *In fact, the temperature stresses in the doubly built-in arch, if it be set up at practically lowest temperature, tend to relieve the stresses due to load.* This is impossible with the two-pivoted arch, for in this case it will be found that if we set up at lowest temperature, so that the temperature stress of ·57 ton in extrados at section 11 subtracts from the load stress of 9·28 tons, then the temperature stress of ·60 ton will add to the intrados stress of 8·81 tons in section 31 and give a maximum stress in all of 9·41 tons.

Hence we conclude that if we set up the arches in the best way, *i.e.*, both at lowest winter temperature, the stresses due to all sources when most disadvantageous will be :—

Two-pivoted arch, 9·41 tons in intrados at slightly less than $\frac{3}{4}$ span from A, the terminal of the heavier loaded half.
Three-pivoted arch, 9·31 tons in intrados at $\frac{1}{4}$ span from A.
Doubly built-in arch, 10·75 tons in intrados at terminal A, but apart from the terminals only a stress of 7·54 tons maximum in the extrados at about $\frac{1}{3}$ span from A.

Thus we see that considering stress as a whole :—

(i.) The high temperature stresses in the doubly built-in arch are not disadvantageous, if the arch be set up at lowest winter temperature.
(ii.) The maximum stress in that arch exceeds by about 1·4 tons per sq. inch that in the pivoted arches, but occurs at the terminals, which can be easily and gracefully strengthened.
(iii.) For the greater part of the arch the stresses are less, and have a maximum of 7·54 tons as compared with the 9·31 or 9·41 tons of the pivoted arches.

(E.) *Vertical Deflections.*

(α.) Deflections due to dead load :—

Two-pivoted arch, maximum deflection = 10 centims. and occurs at the crown.
Three-pivoted arch, maximum deflection = 1·8 centims. and occurs at the crown.
Doubly built-in arch, maximum deflection = 10 centims. and occurs at the crown.

(β.) Deflections due to dead and live loads combined :—

Two-pivoted arch, maximum deflection $= 34$ centims. and occurs at $\frac{7}{24}$ span from A.

Three-pivoted arch, maximum deflection $= 25$ centims.* and occurs at $\frac{1}{3}$ span from A.

Doubly built-in arch, maximum deflection $= 20$ centims. and occurs at $\frac{1}{4}$ span from A.

We see from these results that while there is a distinct advantage in the deflection of the three-pivoted arch over the two-pivoted or doubly built-in arches for dead load—a matter of very small importance—the doubly built-in arch is far better than the doubly-pivoted for the combined loads and sensibly better than the three-pivoted. This will be the more manifest if the deflection curves themselves are examined, for it will be seen that the gain of the built-in arch is by no means confined to the reduced maximum deflection, *the average shift without regard to sign is far less than in the case of the other two arches.* In fact, this arch, as we might have anticipated, is far less "flexible," and therefore "oscillatory," under live load than the other types. This appears an advantage of very considerable importance when the types are compared.

If we take for an arch of the present dimensions and cross-section about half the loading we have put on it, we shall get for the pivoted arches maximum compressive stresses of about 4·7 tons at sections not very easy to strengthen, and in the built-in arch a maximum stress of 5·4 tons at an easily strengthened section, with a stress not greater than 3·8 tons at not easily strengthened sections. All these stresses are quite reasonable working stresses, and there really is not much to choose between them. We shall have a deflection of 17 centims. in the two-pivoted, a deflection of 12·5 centims. in one semi-arch and a rise of 13 centims. in the other semi-arch of the three-pivoted arch, and a deflection of 10 centims. in the doubly built-in arch. These deflections, although apparently considerable, appear small beside those calculated by the French engineers for various live loads on the actual Pont Alexandre III.,† and we cannot help thinking that the small gain of a three-pinned structure in the matter of stress is more than lost by its increased "flexibility."

We should be inclined to sum up then as follows: That so far as the theory of arch structure goes—apart from any difficulty of setting up—there are no marked advantages in the three-pinned type over doubly pivoted or doubly built-in arches. But in the matter of smaller advantages, the fact that the maximum stress is at the terminals, and that the variations of deflection due to live load are sensibly less, seem to commend the doubly built-in type over either pivoted type. Further, if the doubly built-in arch be set up at lowest winter temperature, the at first sight high temperature stresses are not disadvantageous, for they tend to *reduce* the working stresses in the structure at higher than the setting-up temperature. We think, therefore, that it would not be wise to unreservedly accept the Pont Alexandre III. as necessarily the best type under all circumstances; theoretically there is probably a majority of small but sensible advantages to be recorded of the St. Louis Bridge type, and without dogmatising, where so much depends on environment and difficulties of construction, we are inclined to accept the doubly built-in arch as the theoretically best of the three types actually considered in this paper.

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* There is also a rise of 25 centims. to 26 centims. in the semi-arch without live load.

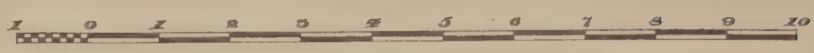
† See the memoirs referred to on p. 43.

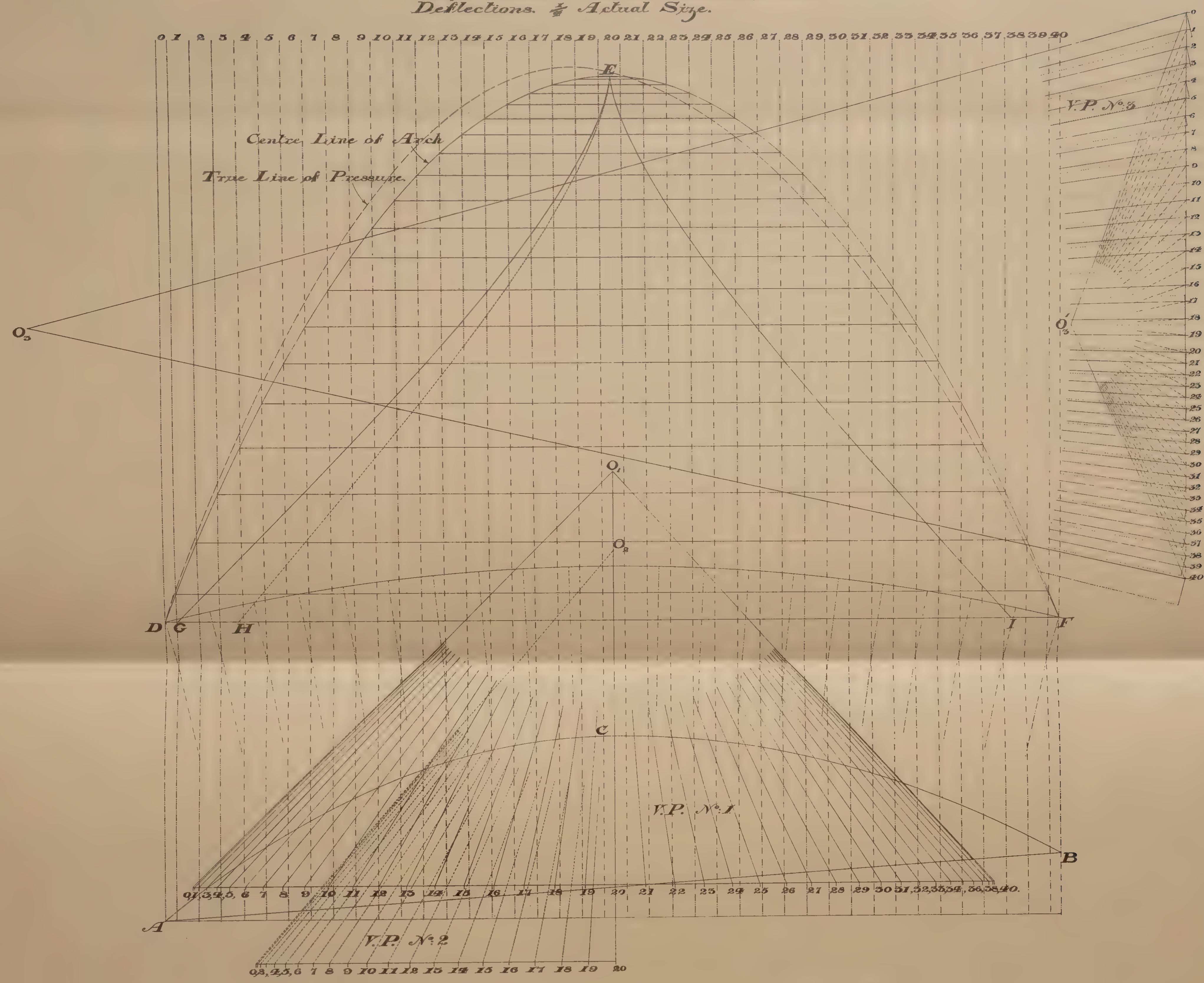
DOUBLY PIVOTED METAL ARCH

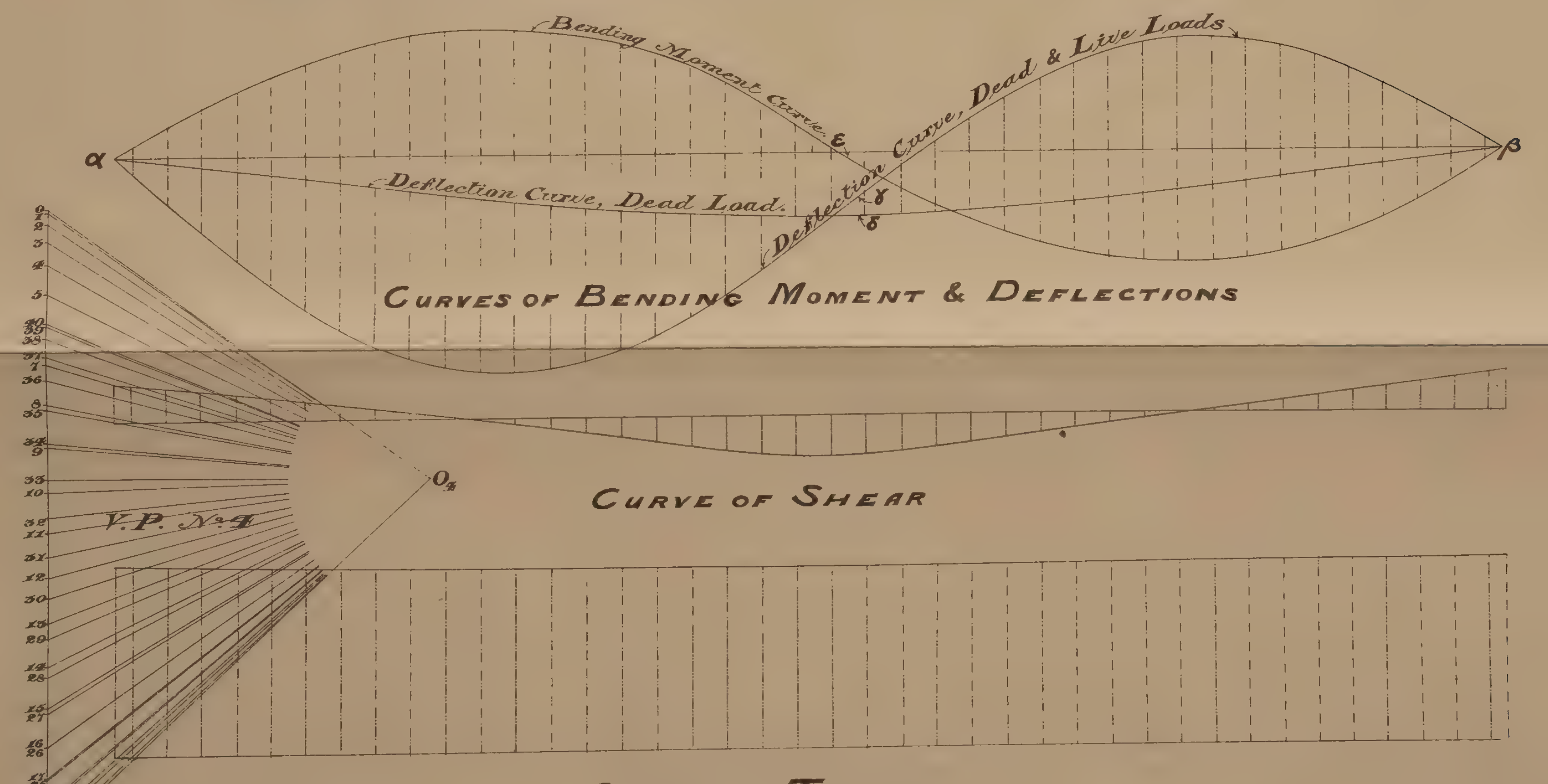
SCALES

Space. 1 inch = 5 metres.
Load. 1 inch = 10 tons.
Shear. 1 inch = 10 tons.
Thrust. 1 inch = 100 tons.
B Mt: 1 inch = 50 metre tons.
Stresses. 1 inch = 5 Tons per sq: inch.
Temp: Stresses. 1 inch = 1 Ton per sq: inch.
Deflections. ⅘ Actual Size.

Original Scale of Inches.






Bending Moment Curve
Deflection Curve, Dead & Live Loads
α
ε
β
Deflection Curve, Dead Load.
γ
δ
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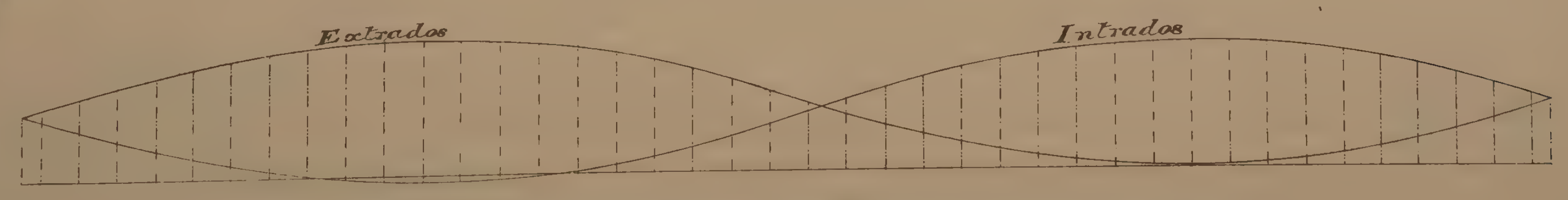
CURVES OF BENDING MOMENT & DEFLECTIONS

CURVE OF SHEAR

CURVE OF THRUST.



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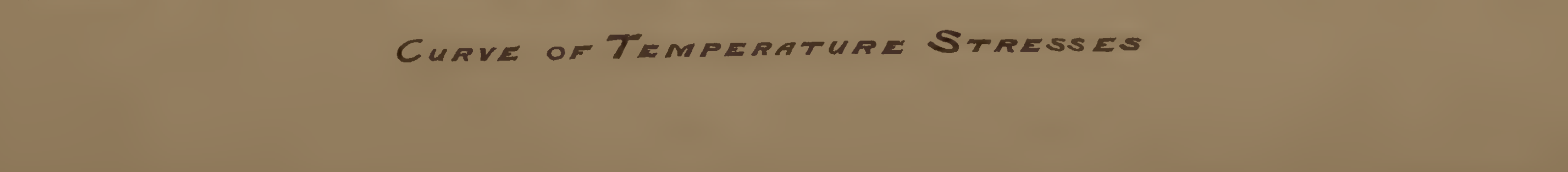


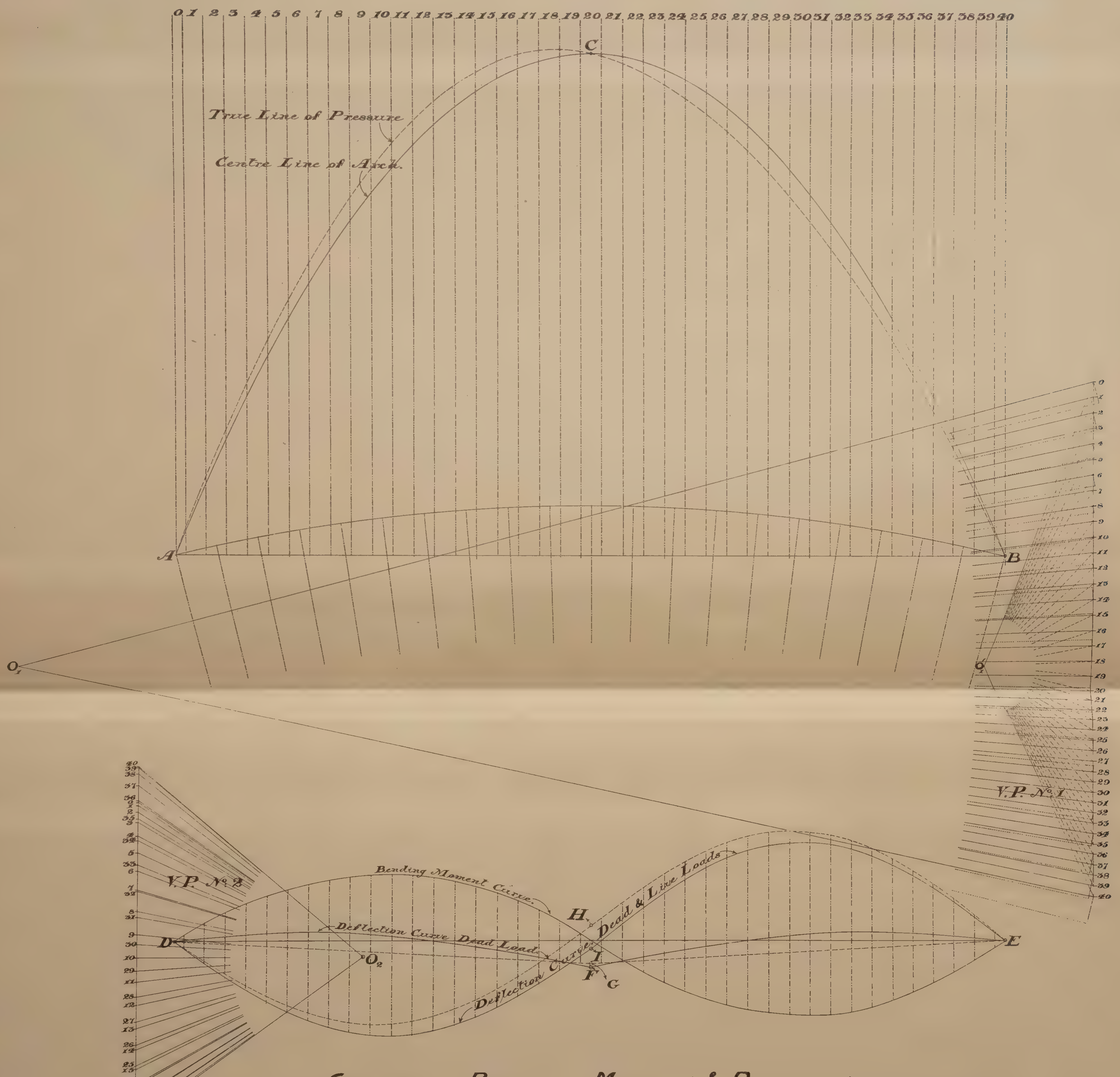
CURVE OF TEMPERATURE STRESSES

TREBLY PIVOTED METAL ARCH

SCALES.

Space. 1 inch = 5 metres.
Load. 1 inch = 10 tons.
Shear. 1 inch = 10 tons.
Thrust. 1 inch = 100 tons.
B Mt: 1 inch = 50 metre tons.
Stresses. 1 inch = 5 tons per sq: inch.
Deflections ¼ Actual Size.
do: Dead Load only. Actual Size

Original Scale of Inches.



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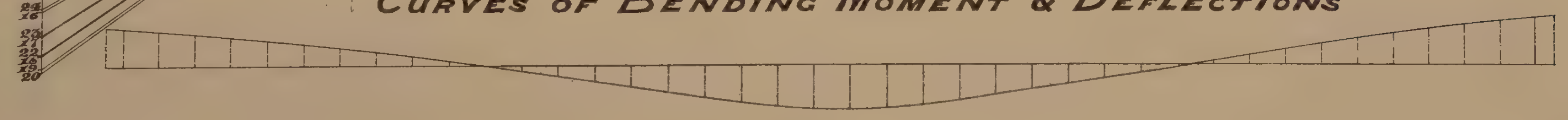

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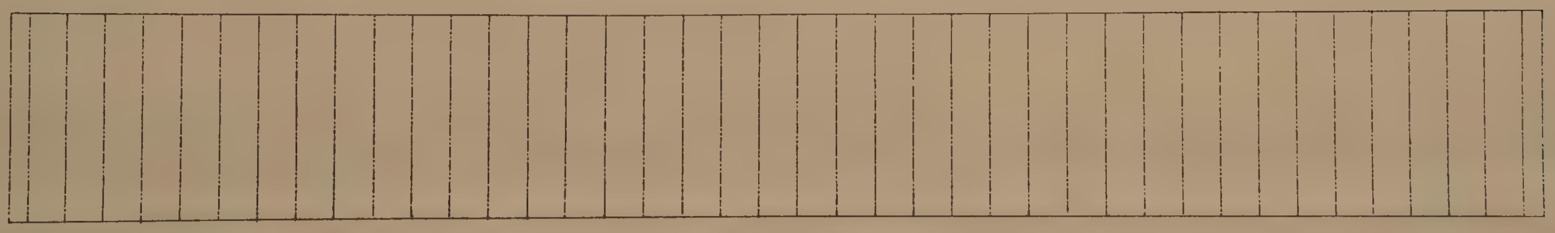
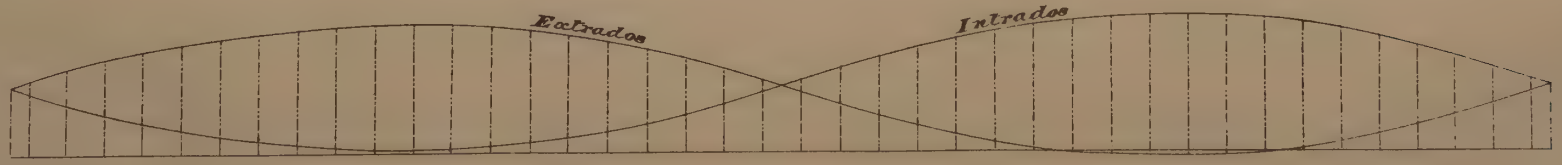


CURVE OF THRUST



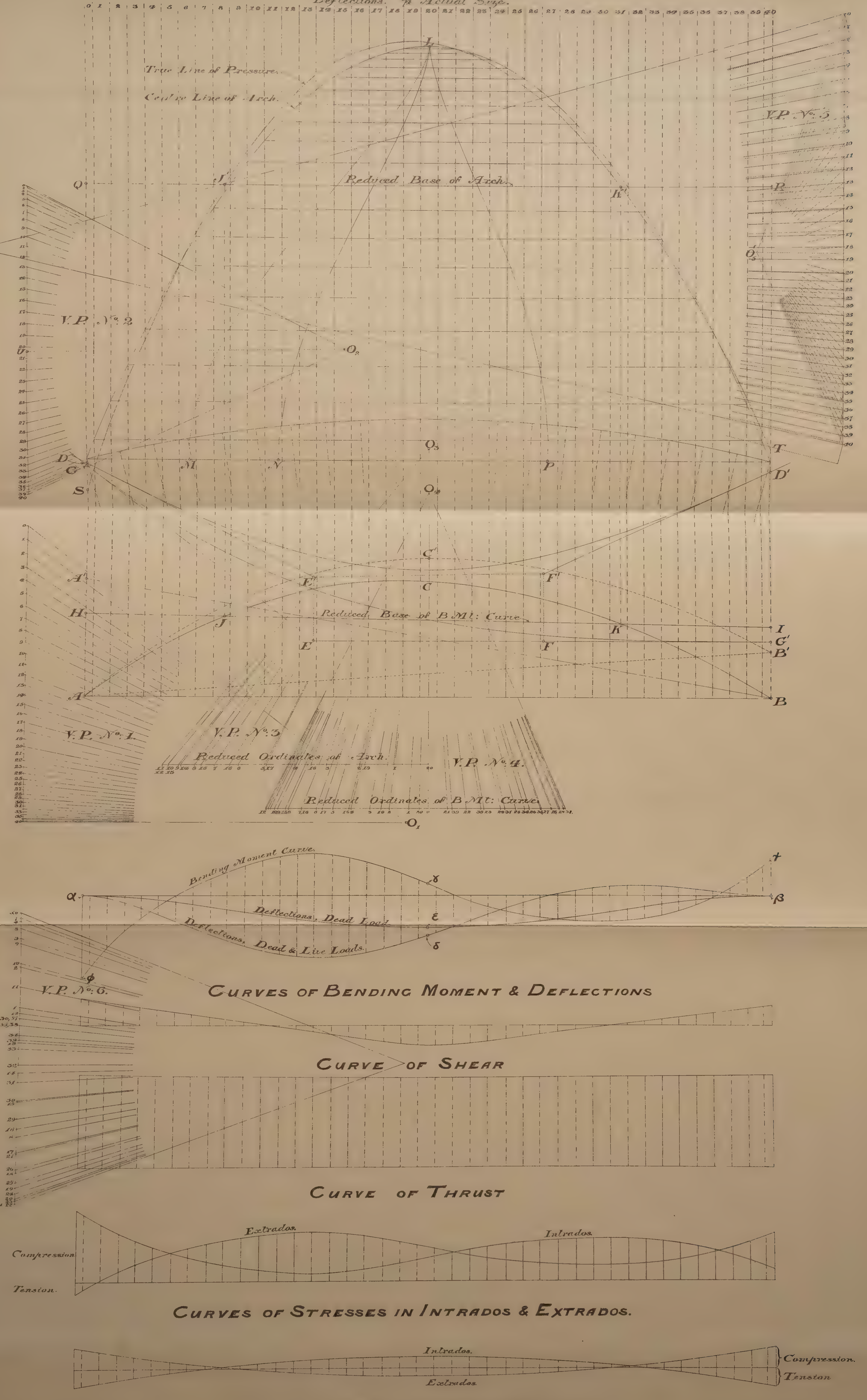
CURVES OF STRESSES IN INTRADOS & EXTRADOS

DOUBLY BUILT-IN METAL ARCH

SCALES

Space. 1 inch = 5 metres.
Load. 1 inch = 10 tons.
Shear. 1 inch = 10 tons.
Thrust. 1 inch = 100 tons.
B. Mt: 1 inch = 50 metre tons.
Stresses. 1 inch = 5 tons per sq: inch.
Temp: Stresses. do: do:
Deflections. ½ Actual Size.

Original Scale of Inches.



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